

THERE IS NO DIOPHANTINE QUINTUPLE

BO HE, ALAIN TOGBÉ, AND VOLKER ZIEGLER

ABSTRACT. A set of m positive integers $\{a_1, a_2, \dots, a_m\}$ is called a Diophantine m -tuple if $a_i a_j + 1$ is a perfect square for all $1 \leq i < j \leq m$. In [16] Dujella proved that there is no Diophantine sextuple and that there are at most finitely many Diophantine quintuples. In particular, a folklore conjecture concerning Diophantine m -tuples states that no Diophantine quintuple exists at all. In this paper we prove this conjecture.

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1. INTRODUCTION

A set of m distinct, positive integers $\{a_1, \dots, a_m\}$ is called a Diophantine m -tuple if $a_i a_j + 1$ is a perfect square for all $1 \leq i < j \leq m$. Diophantus [11] (see also [12]) studied sets of positive rational numbers with the same property, in particular he found the set of four positive rational numbers $\{\frac{1}{16}, \frac{33}{16}, \frac{17}{4}, \frac{105}{16}\}$. But, the first example for an integral Diophantine quadruple $\{1, 3, 8, 120\}$ was found by Fermat. Later Euler showed that there exist infinitely many Diophantine quadruples.

One of the first key results concerning the possible existence of Diophantine quintuples was achieved in 1969, by Baker and Davenport [2] who proved that the

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fourth element 120 in Fermat's quadruple uniquely extends the Diophantine triple $\{1, 3, 8\}$. Thus, they showed that the Diophantine quadruple $\{1, 3, 8, 120\}$ cannot be extended by a fifth positive integer to a Diophantine quintuple. However, no Diophantine quintuple was yet found and it is a folklore conjecture, the so-called *Diophantine quintuple conjecture*, that no Diophantine quintuple exists. In this paper, we give a prove of this conjecture and prove

Theorem 1. *There does not exist a Diophantine quintuple.*

In view of finding Diophantine quintuples or showing that none exists, one of the most important topics is the extensibility and existence of Diophantine m -tuples. For any fixed pair of positive integers a and b such that $ab + 1 = r^2$ is a perfect square, i.e. $\{a, b\}$ is a Diophantine pair, Euler proved that one can always add to $\{a, b\}$ a third element, namely $a + b + 2\sqrt{ab + 1}$, in order to obtain a Diophantine triple of the form

$$(1) \quad \{a, b, a + b + 2r\}.$$

In fact, for every Diophantine pair $\{a, b\}$ there exist infinitely many positive integers c such that $\{a, b, c\}$ is a Diophantine triple. Moreover, Euler observed that adding $4r(a+r)(b+r)$ to the Diophantine triple (1) one obtains a Diophantine quadruple

$$(2) \quad \{a, b, a + b + 2r, 4r(a + r)(b + r)\}$$

therefore proving the existence of infinitely many Diophantine quadruples. We call a Diophantine triple of form (1) an *Euler triple*, or a *regular triple* (see [27]) and a Diophantine quadruple of form (2) an *Euler quadruple*, or *doubly regular quadruple* (see [35]).

Let $\{a, b, c\}$ be a Diophantine triple (not necessarily of form (1)), i.e. $ab + 1 = r^2$, $ac + 1 = s^2$ and $bc + 1 = t^2$ are all perfect squares. In 1979, Arkin, Hoggatt and Strauss [1] noticed that adding

$$\begin{aligned} d_+ &= a + b + c + 2abc + 2\sqrt{(ab + 1)(ac + 1)(bc + 1)} \\ &= a + b + c + 2abc + 2rst \end{aligned}$$

to the Diophantine triple $\{a, b, c\}$ results in a Diophantine quadruple of the form

$$\{a, b, c, a + b + c + 2abc + 2rst\}.$$

Such a quadruple is called a *regular Diophantine quadruple*. Consequently, Euler quadruples are a special case of regular quadruples. Whereas all known Diophantine quadruples are regular, so several authors (see e.g. [1], [19]) were led to an even stronger version of Theorem 1 which is still open.

Conjecture 1. *If $\{a, b, c, d\}$ is a Diophantine quadruple such that $d > \max\{a, b, c\}$, then $d = d_+$.*

After the result of Baker and Davenport on the extensibility of the triple $\{1, 3, 8\}$, the extensibility of further, generalized pairs and triples was studied. All known results positively support Conjecture 1 (see Table 1). In addition, in a series of papers [30, 31, 32] He and Togbé verified Conjecture 1 for triples of the form $\{k, A^2k + 2A, (A + 1)^2k + 2(A + 1)\}$ with two parameters k and A , where

$2 \leq A \leq 10$ or $A \geq 52330$. In particular, they showed that such a triple can be extended only to a regular quadruple. For more results about the regularity of Diophantine pairs and triples, we refer to [14] and [22]. Recently, Fujita and Miyazaki [27] proved that any fixed Diophantine triple can be extended only to a Diophantine quadruple in at most 11 ways by joining a fourth element exceeding the maximal element in the triple.

Pairs / Triples	References
$\{k-1, k+1, 4k\}$	[13]
$\{1, 3\}$	[19]
$\{k-1, k+1\}$	[24, 4]
$\{a, b\}$ with $b < a + 4\sqrt{a}$	[23]
$\{k, 4k \pm 4\}$	[23, 29]
$\{1, b\}$ with $b-1$ be prime	[33]

TABLE 1. Results on the extensibility of Diophantine pairs and triples

Obviously Conjecture 1 implies Theorem 1, but a proof of Conjecture 1 seems to be still out of reach. We do not even know whether there are infinitely many irregular Diophantine quadruples or not. In case of the Diophantine quintuple conjecture, Theorem 1, several researchers obtained several important results. Particularly, the first absolute bound for the size of Diophantine m -tuples was given by Dujella [15], when he showed that $m \leq 8$. In 2004, Dujella [16] proved that there does not exist a Diophantine sextuple. Moreover, Dujella obtained the following result:

Theorem 2. *There are only finitely many Diophantine quintuples.*

A further important step towards a proof of Theorem 1 was made by Fujita [25], who proved the following:

Theorem 3. *If $\{a, b, c, d, e\}$ is a Diophantine quintuple with $a < b < c < d < e$, then $d = d_+$.*

Many researchers believed that a possible route for the settlement of the Diophantine quintuple conjecture, Theorem 1, is to find good estimates for upper bounds of d and good upper bounds for the number of possible quintuples. The following table contains a summary of the progress made towards Theorem 1:

For a more complete account to Diophantine m -tuples and related problems we refer to [17] or Dujella's web page [18].

2. OUTLINE OF THE PROOF

The three new key arguments that lead to the proof of our main result are:

- (1) The definition of an operator on Diophantine triples and their classification.
- (2) The use of sharp lower bounds for linear forms in three logarithms obtained by applying a result due to Mignotte [37] (see also [5, 6]).
- (3) The use of new congruences in the case of Euler quadruples.

The purpose of this section is to give more details to these three key arguments and an outline of the proof of Theorem 1. After stating some auxiliary results in Section 3, we will define in Section 4 the ∂ - and ∂_D -operators on Diophantine triples, where D is a nonnegative integer. The ∂ -operator is defined for non-Euler triples and yields a new triple which is “closer” to the property of being an Euler triple. One of the key results in Section 4 is that if we apply the ∂ operator repeatedly we always arrive at an Euler triple in finite time. This allows us to introduce the *degree* of a Diophantine triple. Roughly speaking an Euler triple has degree 0 and a triple to which an D -fold application of ∂ yields an Euler triple has degree D . This leads us to a new classification of Diophantine triples.

In Section 5, we set up a system of Pell equations associated with a Diophantine quintuple $\{a, b, c, d, e\}$:

$$\begin{aligned} aY^2 - bX^2 &= a - b, \\ aZ^2 - cX^2 &= a - c, \\ bZ^2 - cY^2 &= b - c, \\ aW^2 - dX^2 &= a - d, \\ bW^2 - dY^2 &= b - d, \\ cW^2 - dZ^2 &= c - d. \end{aligned}$$

Using ideas due to Fujita [26], we can show that the solutions to this system of Pell equations satisfy

$$\begin{aligned} Y\sqrt{a} + X\sqrt{b} &= (\sqrt{a} + \sqrt{b})(r + \sqrt{ab})^{2h}, \\ Z\sqrt{a} + X\sqrt{c} &= (\sqrt{a} + \sqrt{c})(s + \sqrt{ac})^{2j}, \\ Z\sqrt{b} + Y\sqrt{c} &= (\sqrt{b} + \sqrt{c})(t + \sqrt{bc})^{2k}, \\ W\sqrt{a} + X\sqrt{d} &= (\varepsilon\sqrt{a} + \sqrt{d})(x + \sqrt{ad})^{2l}, \\ W\sqrt{b} + Y\sqrt{d} &= (\varepsilon\sqrt{b} + \sqrt{d})(y + \sqrt{bd})^{2m}, \\ W\sqrt{c} + Z\sqrt{d} &= (\varepsilon\sqrt{c} + \sqrt{d})(z + \sqrt{cd})^{2n}, \end{aligned}$$

Authors	Year	$d \leq$	# Quintuples
Dujella [16]	2004	10^{2171}	10^{1930}
Fujita [26]	2010	10^{830}	10^{276}
Filipin and Fujita [21]	2013	10^{100}	10^{96}
Elsholtz, Filipin and Fujita [20]	2013	$3.5 \cdot 10^{94}$	$6.8 \cdot 10^{32}$
Wu and He [40]	2014	10^{74}	n/a
Cipu [7]	2015	$10^{72.188}$	10^{31}
Trudgian [39]	2015	$4.02 \cdot 10^{70}$	$2.3 \cdot 10^{29}$
Cipu, Trudgian [8]	2016	$7.228 \cdot 10^{67}$	$5.441 \cdot 10^{26}$

TABLE 2. Upper bounds for d and the number of quintuples

for some nonnegative integers h, j, k, l, m and n . By extending the classical gap principles first introduced by Dujella and Pethő [19], we find several relations and lower bounds for those exponents, most important the inequality $h > 6.2\sqrt{ac}$ (see Lemma 16, in Section 6).

The next step (Section 7) is to find an upper bound for h by using Baker's method, i.e. using lower bounds for linear forms in logarithms. In view of some new results proved in Section 6 we find a slight improvement of the latest result due to Cipu and Trudgian [8]. Using deep results on lower bounds for linear forms in three logarithms due to Mignotte [37] (see also [5, 6]) we can further improve the bound and finally arrive at the upper bounds $d < 1.83 \cdot 10^{52}$ and $h < 5.136 \cdot 10^{13}$. In view of an automatic computer verification these bounds still seem to be too large and some new idea is needed to complete the proof of Theorem 1.

This new idea is presented in Section 8, where we prove new congruences in the case that the Diophantine quintuple $\{a, b, c, d, e\}$ contains an Euler triple. In particular, we show that at least one of the following congruences is satisfied under the hypotheses that $\{a, b, c\}$ is an Euler triple:

- $l \equiv n \equiv 0 \pmod{s}$,
- $m \equiv n \equiv 0 \pmod{t}$,
- $n \equiv \pm r \pmod{st}$.

With an application of Laurent's result [34] on lower bounds for linear forms in two logarithms, the first two congruences yield $s, t < 22023$ respectively (see Lemmas 22 and 23). And the results obtained in Section 7 yield $r < 900154$ provided that $n \equiv \pm r \pmod{st}$. These upper bounds are small enough to use a variant of the Baker-Davenport reduction method. Thus, we may conclude that an Euler triple cannot be extended to a Diophantine quintuple.

In the case that the Diophantine triple $\{a, b, c\}$ is of degree 1, we use again the upper bounds obtained in Section 7 and obtain that $r < 2315167$ and $a < 93596$. These bounds are again small enough to check case by case that no Diophantine quintuple exists with the assistance of a computer.

So we are left to the case that $\{a, b, c\}$ is of degree at least two. Here, we apply the ∂ operator to the triple $\{a, b, c\}$ at least two times which yields a new triple $\{a', b', c'\}$ which is much "smaller" than the original one. By the sharp upper bounds from Section 7 we obtain a feasible number of Diophantine triples that might be extendable to a Diophantine quintuple. Again a computer verification yields that no Diophantine quintuple exists. Thus, also no Diophantine triple of degree > 1 can be extended to a Diophantine quintuple.

Putting these last three results together, we immediately get our main result, i.e. Theorem 1.

3. AUXILIARY RESULTS

For a Diophantine triple $\{a, b, c\}$, we define d_+ and d_- by

$$d_+ = d_+(a, b, c) = a + b + c + 2abc + 2\sqrt{(ab+1)(ac+1)(bc+1)}$$

and

$$d_- = d_-(a, b, c) = a + b + c + 2abc - 2\sqrt{(ab+1)(ac+1)(bc+1)}.$$

Let $ab+1=r^2$, $ac+1=s^2$ and $bc+1=t^2$, then we have

$$ad_{\pm}+1=(rs\pm at)^2, \quad bd_{\pm}+1=(rt\pm bs)^2, \quad cd_{\pm}+1=(cr\pm st)^2.$$

Without loss of generality, assume that $a < b < c$. Then we have the following two Lemmas which will be frequently used in this paper without special reference.

Lemma 1 (Lemma 4 of [28]). *If $\{a, b, c\}$ is a Diophantine triple with $a < b < c$, then $c = a + b + 2r$ or $c > 4ab$.*

Remark 3.1. The statement of Lemma 4 in [28] is slightly different from that given here. In the notation of Jones' paper Lemma 4 of [28] states that $c > 4c'ab$, where c' is some explicitly given quantity and it is easy to show that $c' = 0$ if and only if $c = a + b \pm 2r$.

Lemma 2. $4abc + c < d_+(a, b, c) < 4abc + 4c$.

Remark 3.2. Note that the inequality in Lemma 2 was stated by Dujella [16, page 189]. But, we could not find a proof or reference for this statement. However, for the sake of completeness we give here a short proof of this inequality.

Proof of Lemma 2. The proof of the first inequality is straightforward by noting that

$$\begin{aligned} d_+ &= a + b + c + 2abc + 2\sqrt{(ab+1)(ac+1)(bc+1)} \\ &\geq a + b + c + 2abc + 2\sqrt{(ab)(ac)(bc)} = a + b + c + 4abc \\ &\geq c + 4abc. \end{aligned}$$

Now we turn to the second inequality. By collecting all non-square root terms on the left hand side and taking squares on both sides of the inequality we get the inequality

$$4(ab+1)(ac+1)(bc+1) \leq (2abc + 3c - a - b)^2.$$

After expanding this inequality we are left to prove that

$$8b^2ac + 8a^2bc + 4ab + 4ac + 4bc + 4 \leq 8c^2ab + (3c - a - b)^2$$

holds. By Lemma 1, we have $c \geq a + b + 2r$. Thus we have to check that the inequality

$$4ab + 4ac + 4bc + 4 \leq 16rabc + (3c - a - b)^2$$

holds. However, it is easy to see that

$$4ab + 4ac + 4bc + 4 \leq 4abc + 4abc + 4abc + 4abc \leq 16rabc + (3c - a - b)^2.$$

□

The following two recent results are essential in the proof of Theorem 1 and will be used frequently without any special reference. In particular the inequality $b > 3a$ will be used several times.

Lemma 3 (Theorem 1.1 of [10], see also [9]). *Let $\{a, b, c, d, e\}$ be a Diophantine quintuple with $a < b < c < d < e$. Then $b > 3a$. Moreover, if $c > a + b + 2\sqrt{ab + 1}$ then $b > \max\{24a, 2a^{3/2}\}$.*

Lemma 4 (Theorem 1.5 of [27]). *Let $\{a, b, c, d\}$ be a Diophantine quadruple with $a < b < c < d$. If either $b \geq 2a$ and $c \geq 3.708b^4$ or $b < 2a$ and $c \geq 18.757b^4$, then $d = d_+$.*

Combining these two results, we obtain the following result.

Lemma 5. *Let $\{a, b, c, d, e\}$ be a Diophantine quintuple with $a < b < c < d < e$. Then, we have $ac < 0.927b^3$.*

Proof. Assume that $\{a, b, c, d, e\}$ is a Diophantine quintuple with $a < b < c < d < e$. By Fujita's result [25] (see Theorem 3), the Diophantine quadruple $\{a, b, c, d\}$ is regular. Consider the irregular Diophantine quadruple $\{a, b, d, e\}$. From Lemma 3, we have $b > 3a$. Hence, Lemma 4 provides $d < 3.708b^4$. On the other hand, we have $d = d_+ > 4abc$. Thus, we obtain $4abc < 3.708b^4$ from which we conclude that $ac < 0.927b^3$. \square

In several concrete computations lower bounds for b , c and d are needed. Several authors have excluded many possibilities for a pair (a, b) to be extendable to a Diophantine quintuple. Using recent results due to Cipu and Fujita [9] and Filipin et al. [23], we can show the following.

Lemma 6. *If $\{a, b, c, d, e\}$ is Diophantine quintuple, with $a < b < c < d < e$, then $b \geq 15$, $c \geq 24$ and $d \geq 1520$.*

Proof. Due to a result of Cipu and Fujita [9] (cf. Lemma 3) and a result due to Filipin et al. [23] we may consider only Diophantine quintuples such that $b > 3a$ and $(a, b) \neq (k, 4k \pm 4)$.

First, note that if $r > 15$ then $b > 15$. Therefore, a simple computer search involving all Diophantine pairs (a, b) not excluded by [9] and [23] such that $2 \leq r \leq 15$ shows that the Diophantine pair $\{1, 15\}$ yields the minimal b . Since $a + b + 2r$ strictly increases with a , we deduce that the smallest not excluded c is $1 + 15 + 2\sqrt{1 \cdot 15 + 1} = 24$. Therefore, $\{1, 15, 24\}$ is the smallest Diophantine triple which is possibly extendable to a Diophantine quintuple. Since any Diophantine triple $\{a, b, c\}$ which is extendable to a Diophantine quintuple $\{a, b, c, d, e\}$ with $a < b < c < d < e$ satisfies $d = d_+(a, b, c)$ (cf. Theorem 3), we deduce that $d \geq d_+(1, 15, 24) = 1520$. \square

4. AN OPERATOR ON DIOPHANTINE TRIPLES

In this section we give a classification of Diophantine triples by defining the ∂ -operator between Diophantine triples. We start with a useful classification for Euler triples in terms of d_- .

Proposition 1. *The Diophantine triple $\{a, b, c\}$ is an Euler triple if and only if $d_-(a, b, c) = 0$.*

Proof. If $\{a, b, c\}$ is an Euler triple, then $c = a + b + 2r$, with $r = \sqrt{ab + 1}$. Moreover, we have

$$\sqrt{ac + 1} = a + r, \quad \sqrt{bc + 1} = b + r.$$

Using these identities, we get

$$\begin{aligned} d_-(a, b, c) &= a + b + c + 2abc - 2\sqrt{(ab + 1)(ac + 1)(bc + 1)} \\ &= 2a + 2b + 2r + 2ab(a + b + 2r) - 2r(a + r)(b + r) \\ &= 2a + 2b + 2r + 2a^2b + 2ab^2 + 4abr - 2r(2ab + ar + br + 1) \\ &= 2a + 2b + 2r + 2a^2b + 2ab^2 - 2a(ab + 1) - 2b(ab + 1) - 2r = 0. \end{aligned}$$

On the other hand, assuming that $d_-(a, b, c) = 0$ implies

$$(a + b + c + 2abc)^2 = 4(ab + 1)(ac + 1)(bc + 1).$$

Expanding and simplifying this equation, we get

$$a^2 + b^2 + c^2 - 2ab - 2ac - 2bc = 4.$$

Some further manipulations yield

$$(c - (a + b))^2 = 4(ab + 1) = 4r^2$$

thus

$$c = a + b \pm 2r.$$

Since $c > b > a$ we may omit the “−” case and the triple $\{a, b, c\}$ is indeed an Euler triple. \square

Also the following formulas gathered in the next proposition will be useful. Loosely speaking these formulas show that d_- and d_+ are in some sense the inverse functions of each other.

Proposition 2. *Let $\{a, b, c\}$ be a Diophantine triple with $c = \max\{a, b, c\}$. We have*

$$a = d_-(b, c, d_+(a, b, c)), \quad b = d_-(a, c, d_+(a, b, c)), \quad c = d_-(a, b, d_+(a, b, c)).$$

Moreover, if $\{a, b, c\}$ is not an Euler triple, then we have

$$c = d_+(a, b, d_-(a, b, c)).$$

In particular, $\{a, b, d_-(a, b, c), c\}$ is a regular Diophantine quadruple.

Proof. We fix a and b and consider $d_+(a, b, x) : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ as a function. Let us consider the following equation with the unknown x and with fixed $y \in \mathbb{R}^+$ such that $y > \max\{a, b, x\}$:

$$a + b + x + 2abx + 2\sqrt{(ab + 1)(ax + 1)(bx + 1)} = y.$$

This equation yields the quadratic equation

$$(a + b + x + 2abx - y)^2 = 4(ab + 1)(ax + 1)(bx + 1)$$

and solving for x yields the two solutions

$$x = a + b + y + 2aby + 2\sqrt{(ab + 1)(ay + 1)(by + 1)}$$

and

$$x = a + b + y + 2aby - 2\sqrt{(ab+1)(ay+1)(by+1)}.$$

Obviously, the first solution can be discarded since we assume that $y > x$, while the second solution yields

$$x = d_-(a, b, y) = d_-(a, b, d_+(a, b, x)).$$

Since the formulas for d_+ and d_- are symmetric in a , b and c , we obtain the first three formulas. To obtain the fourth formula we may use a similar trick. The last statement is a direct consequence from the forth formula. Note that we have $c = d_+ = d_+(a, b, d_-(a, b, c))$. \square

To any Diophantine triple $\{a, b, c\}$, we may add d_+ to obtain a regular Diophantine quadruple $\{a, b, c, d_+\}$. In particular, we obtain from the triple $\{a, b, c\}$ three new Diophantine triples $\{a, b, d_+\}$, $\{a, c, d_+\}$ and $\{b, c, d_+\}$ related to $\{a, b, c\}$. From a naive point of view we may consider the triples $\{a, b, d_+\}$, $\{a, c, d_+\}$ and $\{b, c, d_+\}$ to be farther away from being an Euler triple than the original triple $\{a, b, c\}$. Now, let us reverse these observations. Thus, given a non-Euler triple $\{a, b, c\}$ we want to get a new Diophantine triple $\{a', b', c'\}$ that is closer to the property of being an Euler triple. In order to specify these ideas we introduce the ∂ -operator:

Definition. We define ∂ to be an operator which sends a non-Euler triple $\{a, b, c\}$ to a Diophantine triple $\{a', b', c'\}$ such that

$$\partial(\{a, b, c\}) = \{a, b, c, d_-(a, b, c)\} - \{\max(a, b, c)\},$$

where $\{a, b, c, d_-(a, b, c)\} - \{\max(a, b, c)\}$ denotes the set which we obtain by removing the maximal element from the set $\{a, b, c, d_-(a, b, c)\}$.

For a nonnegative integer D , we can define the operator ∂_{-D} on Diophantine triples recursively by

(1) For any Diophantine triple $\{a, b, c\}$ we define

$$\partial_0(\{a, b, c\}) = \{a, b, c\}.$$

(2) Provided that $\partial_{-(D-1)}(\{a, b, c\})$ is not an Euler triple, we recursively define

$$\partial_{-D}(\{a, b, c\}) = \partial(\partial_{-(D-1)}(\{a, b, c\})), \quad \text{for } D \geq 1.$$

Moreover, we put

$$d_{-D}(a, b, c) = d_-(\partial_{-D+1}(\{a, b, c\})).$$

In particular, we have that $\partial = \partial_{-1}$ and

$$\partial_{-2}(\{a, b, c\}) = \partial(\partial_{-1}(\{a, b, c\})).$$

Proposition 3. For any fixed Diophantine triple $\{a, b, c\}$, there exists a unique positive integer $D < \frac{\log(abc)}{\log 12}$ such that $d_{-(D+1)}(a, b, c) = 0$.

Proof. If $\{a, b, c\}$ is an Euler triple, then the result is obtained by Proposition 1.

Now, let us assume that $\{a, b, c\}$ is not an Euler triple. Since by Proposition 2 we know that $\{a, b, d_{-1}(a, b, c), c\}$ is a regular Diophantine quadruple, we deduce that $c > 4ab \cdot d_{-1}(a, b, c)$ by Lemma 2. In particular, we have $ab \cdot d_{-1}(a, b, c) < \frac{c}{4} < \frac{abc}{12}$. Note that $ab \geq 3$. This implies that the product $a'b'c'$ of the elements of the corresponding triple $\{a', b', c'\} := \partial_{-k}(\{a, b, c\})$ is less than $\frac{abc}{12^k}$, provided that the previous $k-1$ images were not Euler triples. Thus, there exists some suitable positive integer D less than $\frac{\log(abc)}{\log 12}$ such that $\{a'', b'', c''\} := \partial_{-D}(\{a, b, c\})$ is an Euler triple and by Proposition 1, we have $d_{-(D+1)}(a, b, c) = 0$.

The uniqueness of D is a direct consequence from the fact that the product $a'b'c'$ with $\{a', b', c'\} := \partial_{-k}(\{a, b, c\})$ is strictly decreasing with k until we get an Euler triple. \square

Definition. We say that a Diophantine triple $\{a, b, c\}$ is of degree D and is generated by an Euler triple $\{a', b', c'\}$, if $d_{-(D+1)}(a, b, c) = 0$ and $\partial_{-D}(\{a, b, c\}) = \{a', b', c'\}$. If the triple $\{a, b, c\}$ is of degree D we simply write $\deg(a, b, c) = D$.

Remark 4.1. Note that in the definition the triple $\{a', b', c'\}$ is an Euler triple due to Proposition 1 since $d_{-}(a', b', c') = 0$ by assumption.

Remark 4.2. Note that there are not too many Diophantine triples of small degree which are generated by a fixed Euler triple $\{a, b, c\}$. Indeed, for an arbitrary but fixed Euler triple $\{a, b, c\}$, there are at most 3^D Diophantine triples generated by $\{a, b, c\}$ with $\deg(a, b, c) = D$.

Proof. The proof will be done by induction on D . If $D = 0$, then $\{a, b, c\}$ is the only Diophantine triple of degree 0 generated by $\{a, b, c\}$. Assume that the result holds for $D = k \geq 0$. Every triple $\{a', b', c'\}$ which is generated by $\{a, b, c\}$ and which is of degree D yields three new triples $\{b', c', d_+(a', b', c')\}$, $\{a', c', d_+(a', b', c')\}$ and $\{a', b', d_+(a', b', c')\}$, which are all of degree $D+1$ and which are sent back to $\{a', b', c'\}$ by the ∂ operator, due to Proposition 2. Thus, we obtain at most $3 \cdot 3^D = 3^{D+1}$ new triples of degree $D+1$. \square

Now, we know the structure of all Diophantine triples which are generated by some Euler triples. For each Euler triple, we get a ternary tree with root $\{a, b, a+b+2r\}$. For example, the triples $\{1, 3, 120\}$, $\{1, 8, 120\}$, $\{3, 8, 120\}$ have degree 1 and are generated by the Fermat triple $\{1, 3, 8\}$.

5. SYSTEM OF PELL EQUATIONS

Let $\{a, b, c\}$ be a Diophantine triple with $a < b < c$, and r, s, t positive integers such that

$$ab + 1 = r^2, \quad ac + 1 = s^2, \quad bc + 1 = t^2.$$

Furthermore, suppose that $\{a, b, c, d, e\}$ is a Diophantine quintuple with $a < b < c < d < e$, and put

$$ad + 1 = x^2, \quad bd + 1 = y^2, \quad cd + 1 = z^2,$$

with positive integers x, y, z . Then, there exist integers X, Y, Z, W such that

$$ae + 1 = X^2, \quad be + 1 = Y^2, \quad ce + 1 = Z^2, \quad de + 1 = W^2.$$

Note that if we fix $d = d_+$, which we may assume due to Fujita's result [25] (cf. Theorem 3), then we have

$$x = at + rs, \quad y = bs + rt, \quad z = cr + st.$$

By eliminating e from the above equations, we obtain the following system of Pell equations:

$$(3) \quad aY^2 - bX^2 = a - b,$$

$$(4) \quad aZ^2 - cX^2 = a - c,$$

$$(5) \quad bZ^2 - cY^2 = b - c,$$

$$(6) \quad aW^2 - dX^2 = a - d,$$

$$(7) \quad bW^2 - dY^2 = b - d,$$

$$(8) \quad cW^2 - dZ^2 = c - d.$$

Let us state the following result concerning Pell equations of the form (3)–(8).

Lemma 7. *Every integer solution to a Pell equation of the form*

$$aY^2 - bX^2 = a - b,$$

with $ab + 1 = r^2$ is obtained from

$$Y\sqrt{a} + X\sqrt{b} = (y_0\sqrt{a} + x_0\sqrt{b})(r + \sqrt{ab})^n,$$

where n , x_0 and y_0 are integers such that $n \geq 0$,

$$1 \leq x_0 \leq \sqrt{\frac{a(b-a)}{2(r-1)}}, \quad \text{and} \quad 1 \leq |y_0| \leq \sqrt{\frac{(r-1)(b-a)}{2a}}.$$

Proof. The Lemma is a direct application of the general theory of Pell equations as described in [38, Theorem 108a] after some small modifications. In particular, this specific case has also been studied by Dujella [15, Lemma 1]. \square

We apply Lemma 7 to the system of Pell equations (3)–(8) and obtain

$$(9) \quad Y\sqrt{a} + X\sqrt{b} = Y_{h'}^{(a,b)}\sqrt{a} + X_{h'}^{(a,b)}\sqrt{b} = (Y_0\sqrt{a} + X_0\sqrt{b})(r + \sqrt{ab})^{h'},$$

$$(10) \quad Z\sqrt{a} + X\sqrt{c} = Z_{j'}^{(a,c)}\sqrt{a} + X_{j'}^{(a,c)}\sqrt{c} = (Z_1\sqrt{a} + X_1\sqrt{c})(s + \sqrt{ac})^{j'},$$

$$(11) \quad Z\sqrt{b} + Y\sqrt{c} = Z_{k'}^{(b,c)}\sqrt{b} + Y_{k'}^{(b,c)}\sqrt{c} = (Z_2\sqrt{b} + Y_2\sqrt{c})(t + \sqrt{bc})^{k'},$$

$$(12) \quad W\sqrt{a} + X\sqrt{d} = W_{l'}^{(a,d)}\sqrt{a} + X_{l'}^{(a,d)}\sqrt{d} = (W_3\sqrt{a} + X_3\sqrt{d})(x + \sqrt{ad})^{l'},$$

$$(13) \quad W\sqrt{b} + Y\sqrt{d} = W_{m'}^{(b,d)}\sqrt{b} + Y_{m'}^{(b,d)}\sqrt{d} = (W_4\sqrt{b} + Y_4\sqrt{d})(y + \sqrt{bd})^{m'},$$

$$(14) \quad W\sqrt{c} + Z\sqrt{d} = W_{n'}^{(c,d)}\sqrt{c} + Z_{n'}^{(c,d)}\sqrt{d} = (W_5\sqrt{c} + Z_5\sqrt{d})(z + \sqrt{cd})^{n'},$$

for some nonnegative integers h', j', k', l', m', n' and integers $Y_0, X_0, Z_1, X_1, Z_2, Y_2, W_3, X_3, W_4, Y_4, W_5, Z_5$.

In view of the relations (12), (13), and (14) we have that $W = W_{l'}^{(a,d)} = W_{m'}^{(b,d)} = W_{n'}^{(c,d)}$, where $W_{l'}^{(a,d)}$, $W_{m'}^{(b,d)}$, and $W_{n'}^{(c,d)}$ satisfy the following recursions:

$$\begin{aligned} W_0^{(a,d)} &= W_3, & W_1^{(a,d)} &= xW_3 + dX_3, & W_{l'+2}^{(a,d)} &= 2xW_{l'+1}^{(a,d)} - W_{l'}^{(a,d)}, \\ W_0^{(b,d)} &= W_4, & W_1^{(b,d)} &= yW_4 + dY_4, & W_{m'+2}^{(b,d)} &= 2yW_{m'+1}^{(b,d)} - W_{m'}^{(b,d)}, \\ W_0^{(c,d)} &= W_5, & W_1^{(c,d)} &= zW_5 + dZ_5, & W_{n'+2}^{(c,d)} &= 2zW_{n'+1}^{(c,d)} - W_{n'}^{(c,d)}. \end{aligned}$$

The next two lemmas will help us to better understand the structure of the solutions to the system of Pell equations (3)–(8). In particular, the next lemma due to Fujita [26] takes care of the subsystem of Pell equations (6)–(8)

Lemma 8 ([26], Lemma 2.2). *If $W = W_{l'}^{(a,d)} = W_{m'}^{(b,d)} = W_{n'}^{(c,d)}$, then we have*

$$l' \equiv m' \equiv n' \equiv 0 \pmod{2}$$

and

$$W_3 = W_4 = W_5 = \varepsilon = \pm 1.$$

Our next aim is to extend Fujita's result stated in Lemma 8.

Lemma 9. *We have*

$$h' \equiv j' \equiv k' \equiv 0 \pmod{2}$$

and

$$X_0 = X_1 = Y_0 = Y_2 = Z_1 = Z_2 = 1.$$

Proof. First we consider the subsystem of Pell equations (3) and (7), i.e. the system

$$(15) \quad \begin{aligned} aY^2 - bX^2 &= a - b, \\ bW^2 - dY^2 &= b - d. \end{aligned}$$

In particular we are interested in the first equation of system (15). By Lemma 7, the integer solutions to the first equation of (15) are obtained by

$$Y\sqrt{a} + X\sqrt{b} = (Y_0\sqrt{a} + X_0\sqrt{b})(r + \sqrt{ab})^{h'}, \quad h' \geq 0,$$

where $|Y_0| \leq \sqrt{\frac{(r-1)(b-a)}{2a}} < \sqrt{\frac{b\sqrt{b}}{2\sqrt{a}}} < 0.71b^{3/4}$. Further, by (9) we may write $Y = Y_{h'}^{(a,b)}$, where $Y_{h'}^{(a,b)}$ satisfies the following recursion

$$Y_0^{(a,b)} = Y_0, \quad Y_1^{(a,b)} = rY_0 + bX_0, \quad Y_{h'+2}^{(a,b)} = 2rY_{h'+1}^{(a,b)} - Y_{h'}^{(a,b)}.$$

Thus, we get

$$(16) \quad Y_{h'}^{(a,b)} \equiv \begin{cases} Y_0 \pmod{b}, & \text{if } h' \text{ even,} \\ Y_0r \pmod{b}, & \text{if } h' \text{ odd.} \end{cases}$$

Next we consider the second equation of system (15). By Lemma 8, we know that $W_4 = \varepsilon = \pm 1$. This implies together with Pell equation (7) that $Y_4 = \pm 1$. Note that due to Lemma 7, we may discard the case that $Y_4 = -1$ and therefore

we may assume that $Y_4 = 1$. Using (13), we have $Y = Y_{m'}^{(b,d)}$ and we obtain the following recursion:

$$Y_0^{(b,d)} = 1, \quad Y_1^{(b,d)} = y + \varepsilon b, \quad Y_{m'+2}^{(b,d)} = 2yY_{m'+1}^{(b,d)} - Y_{m'}^{(b,d)}.$$

From this recursion we deduce that

$$(17) \quad Y_{m'}^{(b,d)} \equiv 1 \pmod{b}, \quad \text{for even } m'.$$

Note that due to Lemma 8 we may assume that m' is even. If $Y_{h'}^{(a,b)} = Y_{m'}^{(b,d)}$ with h' odd, then from the congruences (16) and (17) we have that $Y_0 r \equiv 1 \pmod{b}$. Multiplying both sides by r , we obtain $Y_0 \equiv r \pmod{b}$. The bound $|Y_0| < 0.71b^{3/4}$ implies $Y_0 = r$ or $Y_0 = r - b$. Let us consider the case $Y_0 = r$ first. As (Y_0, X_0) is an integer solution to the first equation in (15) we obtain

$$X_0^2 = \frac{aY_0^2 - a + b}{b} = \frac{a(ab + 1) - a + b}{b} = a^2 + 1.$$

But $X_0^2 = a^2 + 1$ implies that $X_0 = 1$ and $a = 0$, which is a contradiction.

Now, assume that $Y_0 = r - b$. Since $b > 3a$ due to Lemma 3 we have $a \leq b/3 - 1$, hence

$$(18) \quad 0.71b^{3/4} > b - r \geq b - \sqrt{(b/3 - 1)b + 1} \geq b \left(1 - \frac{1}{\sqrt{3}}\right) > 0.422b.$$

Therefore, we obtain $b \leq 8.1 < 15$, which is a contradiction to Lemma 6.

Hence, if $Y = Y_{h'}^{(a,b)} = Y_{m'}^{(b,d)}$, then h' is even. Once again, from the congruences (16) and (17), we obtain that $Y_0 \equiv 1 \pmod{b}$. Since $|Y_0| < 0.71b^{3/4}$, we have $Y_0 = 1$. Thus, we get $X_0 = \pm 1$. Since Lemma 7 we may assume that X_0 is positive and we obtain $X_0 = Y_0 = 1$.

Similarly, by replacing Y by Z and b by c we obtain from the system of Pell equations (15) the system

$$\begin{aligned} aZ^2 - cX^2 &= a - c, \\ cW^2 - dZ^2 &= c - d. \end{aligned}$$

Now, applying the same arguments as above, we deduce that j' is even and $X_1 = Z_1 = 1$. In particular, in this case we have to exclude the solution $Z_1 = s - c$. However in this case, instead of inequality (18) we obtain the inequality

$$0.71c^{3/4} > |Z_1| = c - s \geq c - \sqrt{(c/3 - 1)c + 1} \geq c \left(1 - \frac{1}{\sqrt{3}}\right) > 0.422c$$

and deduce that j' is even.

The same method also works if we replace Y by Z , X by Y , a by b and b by c in the system of Pell equations (15), i.e if we consider the system

$$\begin{aligned} bZ^2 - cY^2 &= b - c, \\ cW^2 - dZ^2 &= c - d. \end{aligned}$$

In this case, we achieve that k' is even and $Y_2 = Z_2 = 1$. Thus, the only non straightforward step is to exclude the fundamental solution $Z_2 = t - c$. If $\{a, b, c\}$

is not an Euler triple, then we have $c \geq 4ab$ and in particular $b \leq c/4$. Therefore, instead of inequality (18) we obtain the inequality

$$0.71c^{3/4} > |Z_2| = c - t \geq c - \sqrt{\frac{c^2}{4} + 1} > 0.499c,$$

which yields a contradiction to the fact that $c \geq 24$. Also note that the last inequality is due to $c \geq 24$. Therefore, let us assume that $c = a + b + 2r$. Then, we obtain

$$\begin{aligned} s = c - t = |Z_2| &\leq \sqrt{\frac{(t-1)(c-b)}{2b}} \\ &< \sqrt{\frac{\sqrt{bc}(a+b+2r-b)}{2b}} < \sqrt{\sqrt{c/b}(a/2+r)} < \sqrt{s \cdot s} = s, \end{aligned}$$

which is a contradiction. Hence, in any case we obtain that k' is even. \square

Furthermore, due to Lemma 7 we deduce that $X_3 = Y_4 = Z_5 = 1$ since $W_3 = W_4 = W_5 = \varepsilon = \pm 1$. So from now and on, we may write

$$h' = 2h, \quad j' = 2j, \quad k' = 2k, \quad l' = 2l, \quad m' = 2m, \quad n' = 2n,$$

where h, j, k, l, m, n are positive integers. Note that we may assume that the exponents are positive because a vanishing exponent would yield that either X , Y or Z is one, thus one of a, b, c, d, e is zero. Finally, we may rewrite the formulas (9)-(14) to

$$(19) \quad Y\sqrt{a} + X\sqrt{b} = (\sqrt{a} + \sqrt{b})(r + \sqrt{ab})^{2h},$$

$$(20) \quad Z\sqrt{a} + X\sqrt{c} = (\sqrt{a} + \sqrt{c})(s + \sqrt{ac})^{2j},$$

$$(21) \quad Z\sqrt{b} + Y\sqrt{c} = (\sqrt{b} + \sqrt{c})(t + \sqrt{bc})^{2k},$$

$$(22) \quad W\sqrt{a} + X\sqrt{d} = (\varepsilon\sqrt{a} + \sqrt{d})(x + \sqrt{ad})^{2l},$$

$$(23) \quad W\sqrt{b} + Y\sqrt{d} = (\varepsilon\sqrt{b} + \sqrt{d})(y + \sqrt{bd})^{2m},$$

$$(24) \quad W\sqrt{c} + Z\sqrt{d} = (\varepsilon\sqrt{c} + \sqrt{d})(z + \sqrt{cd})^{2n}.$$

6. THE GAP PRINCIPLE AND THE CLASSICAL CONGRUENCE

If there exists a positive integer e such that the Diophantine quadruple $\{a, b, c, d\}$ can be extended to a quintuple $\{a, b, c, d, e\}$ with $a < b < c < d < e$, then relations (19)–(24) are fulfilled with positive integers (i, j, k, l, m, n) . Moreover, there are 12 sequences associated with X, Y, Z , and W , with indices i, j, k, l, m , and n as defined in (9)–(14). For example, we have

$$Y_h^{(a,b)}\sqrt{a} + X_h^{(a,b)}\sqrt{b} := (\sqrt{a} + \sqrt{b})(r + \sqrt{ab})^h,$$

and so on. Thus, each of the variables X, Y, Z , and W corresponds to three sequences as follows:

$$\begin{aligned} X &= X_{2h}^{(a,b)} = X_{2j}^{(a,c)} = X_{2l}^{(a,d)}, & Y &= Y_{2h}^{(a,b)} = Y_{2k}^{(b,c)} = Y_{2m}^{(b,d)}, \\ Z &= Z_{2j}^{(a,c)} = Z_{2k}^{(b,c)} = Z_{2n}^{(c,d)}, & W &= W_{2l}^{(a,d)} = W_{2m}^{(b,d)} = W_{2n}^{(c,d)}. \end{aligned}$$

Several authors proved various relations between the indices of these sequences. Let us recall two results due to Dujella [16] and Fujita [26] respectively.

Lemma 10 (Lemma 3 of [16]). *If $Z = Z_{2j}^{(a,c)} = Z_{2k}^{(b,c)}$, then $k - 1 \leq j \leq 2k + 1$.*

Lemma 11 (Lemma 2.3 of [26]). *If $W = W_{2l}^{(a,d)} = W_{2m}^{(b,d)} = W_{2n}^{(c,d)}$, then $4 \leq n \leq m \leq l \leq 2n$.*

Lemmas 10 and 11 reveal relations between the indices k, j, m , and l . In order to get a better understanding of the relations between m, n , and h we prove the following two lemmas.

Lemma 12. *We have $2l \leq 3m$ and $m < l$ unless $m = 0$.*

Proof. From (22) and (23) respectively (12) and (13), we get the recursions

$$\begin{aligned} W_0^{(a,d)} &= \varepsilon, & W_1^{(a,d)} &= \varepsilon x + d, & W_{l+2}^{(a,d)} &= 2xW_{l+1}^{(a,d)} - W_l^{(a,d)}, & l &\geq 0, \\ W_0^{(b,d)} &= \varepsilon, & W_1^{(b,d)} &= \varepsilon y + d, & W_{m+2}^{(b,d)} &= 2yW_{m+1}^{(b,d)} - W_m^{(b,d)}, & m &\geq 0. \end{aligned}$$

Furthermore, solving these recursions explicitly we get

$$\begin{aligned} W_{2l}^{(a,d)} &= \frac{d + \varepsilon\sqrt{ad}}{2\sqrt{ad}} \left(x + \sqrt{ad}\right)^{2l} + \frac{-d + \varepsilon\sqrt{ad}}{2\sqrt{ad}} \left(x - \sqrt{ad}\right)^{2l} \\ W_{2m}^{(b,d)} &= \frac{d + \varepsilon\sqrt{bd}}{2\sqrt{bd}} \left(y + \sqrt{bd}\right)^{2m} + \frac{-d + \varepsilon\sqrt{bd}}{2\sqrt{bd}} \left(y - \sqrt{bd}\right)^{2m}. \end{aligned}$$

We prove that $2l \leq 3m$ first. By considering the intersection of the two recursions $W_{2l}^{(a,d)}$ and $W_{2m}^{(b,d)}$, i.e. the equation $W_{2l}^{(a,d)} = W_{2m}^{(b,d)}$, we get the inequality

$$\begin{aligned} \frac{d + \varepsilon\sqrt{ad}}{2\sqrt{ad}} \left(x + \sqrt{ad}\right)^{2l} - \frac{d + \sqrt{ad}}{2\sqrt{ad}} &< W_{2l}^{(a,d)} = W_{2m}^{(b,d)} \\ &< \frac{d + \varepsilon\sqrt{bd}}{2\sqrt{bd}} \left(y + \sqrt{bd}\right)^{2m} + \frac{-d + \sqrt{bd}}{2\sqrt{bd}}. \end{aligned}$$

The inequality above holds since $0 < x - \sqrt{ad}$, $y - \sqrt{bd} < 1$. We add $\frac{d+\sqrt{ad}}{2\sqrt{ad}}$ to both sides of the inequality and multiply by $\frac{2\sqrt{ad}}{d+\varepsilon\sqrt{ad}}$ afterwards. Then, we get

$$\begin{aligned} (x + \sqrt{ad})^{2l} &< \frac{\sqrt{a}}{\sqrt{b}} \cdot \frac{d + \varepsilon\sqrt{bd}}{d + \varepsilon\sqrt{ad}} (y + \sqrt{bd})^{2m} + \frac{d(\sqrt{b} - \sqrt{a}) + 2\sqrt{abd}}{\sqrt{b}(d + \varepsilon\sqrt{ad})} \\ &< \frac{d + 3\sqrt{ad}}{d - \sqrt{ad}} (y + \sqrt{bd})^{2m}. \end{aligned}$$

Note that

$$\begin{aligned} \frac{d + 3\sqrt{ad}}{d - \sqrt{ad}} &= 1 + \frac{4\sqrt{ad}}{d - \sqrt{ad}} = 1 + \frac{4}{\sqrt{d/a} - 1} \\ &< 1 + \frac{4}{\sqrt{4bc} - 1} < 1 + \frac{4}{\sqrt{1440} - 1} < 10/9 \end{aligned}$$

since $d = d_+ > 4abc$, $b \geq 15$, and $c \geq 24$ (cf. Lemma 6).

Assume that $2l \geq 3m+1$. Then, we have that $(x+\sqrt{ad})^{3m+1} < \frac{10}{9}(y+\sqrt{bd})^{2m}$. Since $x+\sqrt{ad} > 10/9$, we get $(x+\sqrt{ad})^{3m} < (y+\sqrt{bd})^{2m}$ and furthermore

$$(x+\sqrt{ad})^3 < (y+\sqrt{bd})^2.$$

From $x+\sqrt{ad} > 2\sqrt{ad}$ and $y+\sqrt{bd} < 2.1\sqrt{bd}$ (note that $b \geq 15$ and $d \geq 1520$), we have

$$64a^3d^3 = (4ad)^3 < (4.41bd)^2 < 20b^2d^2$$

which yields

$$3.2a^3d < b^2.$$

But this is a contradiction to the fact that $d = d_+ > 4abc > 4ab^2$. Therefore, we have $2l < 3m+1$ and deduce $2l \leq 3m$.

The proof that $m < l$ is similar. By Lemma 11 we know that $m \leq l$. Therefore, we may assume for the moment that $m = l$. Thus, we consider the equation $W_{2m}^{(a,d)} = W_{2m}^{(b,d)}$ and obtain the inequality

$$\begin{aligned} \frac{d + \varepsilon\sqrt{ad}}{2\sqrt{ad}} (x + \sqrt{ad})^{2m} + \frac{-d + \sqrt{ad}}{2\sqrt{ad}} &> W_{2m}^{(a,d)} = W_{2m}^{(b,d)} \\ &> \frac{d + \varepsilon\sqrt{bd}}{2\sqrt{bd}} (y + \sqrt{bd})^{2m} - \frac{d + \sqrt{bd}}{2\sqrt{bd}}. \end{aligned}$$

By some similar manipulations as above we obtain

$$(y + \sqrt{bd})^{2m} < \frac{d + 3\sqrt{bd}}{d - \sqrt{bd}} (x + \sqrt{ad})^{2m}.$$

Since

$$\begin{aligned} \frac{d + 3\sqrt{bd}}{d - \sqrt{bd}} &= 1 + \frac{4\sqrt{bd}}{d - \sqrt{bd}} = 1 + \frac{4}{\sqrt{d/b} - 1} \\ &< 1 + \frac{4}{\sqrt{4ac} - 1} < 1 + \frac{4}{\sqrt{96} - 1} < 3/2 \end{aligned}$$

we get

$$(25) \quad \left(\frac{y + \sqrt{bd}}{x + \sqrt{ad}} \right)^{2m} < \frac{3}{2}.$$

On the other hand we have

$$\frac{y + \sqrt{bd}}{x + \sqrt{ad}} > \frac{2\sqrt{bd}}{\sqrt{ad} + 1 + \sqrt{ad}} > 0.999\sqrt{\frac{b}{a}} > 0.999\sqrt{3} > 3/2.$$

Note that $d \geq 1520$ by Lemma 6 and $\frac{b}{a} > 3$ by Lemma 3. Therefore inequality (25) holds only if $m = l = 0$, i.e. we have that $m < l$ unless $m = 0$. \square

Lemma 13. *We have $h \geq 2m$.*

Proof. We proceed similarly as in the proof of Lemma 12. In this case, we consider the equation $Y = Y_{2h}^{(a,b)} = Y_{2m}^{(b,d)}$. From (19) and (23), we obtain the recursions

$$\begin{aligned} Y_0^{(a,b)} &= 1, & Y_1^{(a,b)} &= r + b, & Y_{h+2}^{(a,b)} &= 2rY_{h+1}^{(a,b)} - Y_h^{(a,b)}, & h &\geq 0, \\ Y_0^{(b,d)} &= 1, & Y_1^{(b,d)} &= y + \varepsilon b, & Y_{m+2}^{(b,d)} &= 2yY_{m+1}^{(b,d)} - Y_m^{(b,d)}, & m &\geq 0. \end{aligned}$$

Solving the first recursion, we get

$$Y_{2h}^{(a,b)} = \frac{b + \sqrt{ab}}{2\sqrt{ab}}(r + \sqrt{ab})^{2h} - \frac{b - \sqrt{ab}}{2\sqrt{ab}}(r - \sqrt{ab})^{2h}$$

and deduce that

$$Y_{2h}^{(a,b)} < \frac{1}{2} \left(1 + \sqrt{\frac{b}{a}} \right) (r + \sqrt{ab})^{2h}.$$

Solving the second recursion, we obtain

$$Y_{2m}^{(b,d)} = \frac{\varepsilon b + \sqrt{bd}}{2\sqrt{bd}}(y + \sqrt{bd})^{2m} + \frac{\sqrt{bd} - \varepsilon b}{2\sqrt{bd}}(y - \sqrt{bd})^{2m},$$

which yields

$$Y_{2m}^{(b,d)} > \frac{\sqrt{bd} - b}{2\sqrt{bd}}(y + \sqrt{bd})^{2m} > \frac{1}{2}(1 - \sqrt{b/d})(y + \sqrt{bd})^{2m}.$$

If $Y_{2h}^{(a,b)} = Y_{2m}^{(b,d)}$, then we have

$$(y + \sqrt{bd})^{2m} < \frac{1 + \sqrt{\frac{b}{a}}}{1 - \sqrt{\frac{b}{d}}}(r + \sqrt{ab})^{2h} < (r + \sqrt{ab})^{2h+1}$$

since

$$\frac{1 + \sqrt{\frac{b}{a}}}{1 - \sqrt{\frac{b}{d}}} = 1 + \frac{2}{\sqrt{d/b} - 1} < 1 + \frac{2}{\sqrt{4ac} - 1} < 2 < r + \sqrt{ab}.$$

Note that $ac \geq 24$ in any case.

We claim that $y + \sqrt{bd} > (r + \sqrt{ab})^2$ and deduce from this claim that

$$(r + \sqrt{ab})^{4m} < (r + \sqrt{ab})^{2h+1},$$

which shows that $4m < 2h + 1$. Thus we get $4m \leq 2h$, hence $h \geq 2m$.

Therefore, we are left to justify our claim. In order to show the claim, it suffices to prove that

$$y + \sqrt{bd} > 2\sqrt{bd} > \left(2\sqrt{\frac{4}{3}ab} \right)^2 > \left(2\sqrt{ab+1} \right)^2 > \left(r + \sqrt{ab} \right)^2.$$

The only non obvious inequality is the second one. Squaring both sides yields $bd > \frac{64}{9}a^2b^2$. Since $d > 4abc$, we have to show that $ac > \frac{16}{9}a^2$, which is true since $c > b > 3a$ (cf. Lemma 3). \square

Next, let us state the following useful observation.

Lemma 14. *Let $\{a, b, c, d, e\}$ be a Diophantine quintuple, then we have*

$$(26) \quad al^2 + \varepsilon xl \equiv bm^2 + \varepsilon ym \pmod{4d}.$$

Proof. The congruence above is a direct consequence of a result due to Dujella [15, Lemma 4] in which several congruence relations between the indices were shown. In order to obtain the result, we apply [15, Lemma 4] to the Diophantine quadruple $\{a, b, d, e\}$ and note that since $\{a, b, d, e\}$ is part of a Diophantine quintuple we may consider only the even cases of [15, Lemma 4] (i.e. part (1) and (3) of [15, Lemma 4]) and due to Lemmas 8 and 9 we obtain congruence (26). \square

Most researchers studying Diophantine quintuples used similar congruences to discuss lower bounds for various indices. In [40], Wu and the first author got a strong lower bound for m , namely $m \geq 0.48\sqrt{d/b}$. A slight improvement of the constant from 0.48 to 0.5 was achieved by Cipu [7]. An even better lower bound was given by Cipu and Trudgian in [8]. However we surpass these bounds by proving the following lemma.

Lemma 15. *If $W_{2l}^{(a,d)} = W_{2m}^{(b,d)}$, then $m \geq \left(\frac{\sqrt{17}-1}{2}\right)\sqrt{d/b}$.*

Proof. We consider congruence relation (26) and assume for the moment that

$$(27) \quad al^2 + \varepsilon xl = bm^2 + \varepsilon ym,$$

then we obtain

$$al^2 - bm^2 = \varepsilon(ym - xl).$$

This implies that

$$\begin{aligned} (ym + xl)(al^2 - bm^2) &= \varepsilon(y^2m^2 - x^2l^2) \\ &= \varepsilon((bd + 1)m^2 - (ad + 1)l^2) \\ &= \varepsilon(d(bm^2 - al^2) + m^2 - l^2). \end{aligned}$$

Collecting terms and taking absolute values results in

$$|l^2 - m^2| = |(d + \varepsilon(ym + xl))(bm^2 - al^2)|.$$

If $bm^2 - al^2 = 0$ or $d + \varepsilon(ym + xl) = 0$, then $l = m$. Due to Lemma 12 we deduce that $l = m = 0$, which is impossible. Hence,

$$|l^2 - m^2| \geq |bm^2 - al^2|$$

and we obtain the inequality

$$\left|\frac{b}{a} - \frac{l^2}{m^2}\right| \leq \frac{|l^2 - m^2|}{am^2} = \frac{|(l/m)^2 - 1|}{a}.$$

Using Lemma 12, we have $(l/m)^2 < 2.25$ and by Lemma 3 we get $b/a > 3$. Thus, we obtain

$$(28) \quad 0.75 = 3 - 2.25 \leq \left|\frac{b}{a} - \frac{l^2}{m^2}\right| \leq \frac{|(l/m)^2 - 1|}{a} < \frac{1.25}{a},$$

i.e. $a < \frac{5}{3}$. Therefore, we only need to consider the case that $a = 1$. When $a = 1$ and $b \geq 15$ (cf. Lemma 6), then inequality (28) is impossible. Therefore, relation (27) does not hold.

In the case that equation (27) does not hold, the left side and the right side of (27) differ at least by $4d$. Therefore, we get the inequality

$$4d \leq |bm^2 - al^2 + \varepsilon(ym - xl)| \leq |bm^2 - al^2| + |ym - xl| < bm^2 + ym.$$

Thus, we have

$$(29) \quad 4d \leq bm^2 + ym - 1 = bm^2 + m\sqrt{bd} + \frac{m}{\sqrt{bd+1} + \sqrt{bd}} - 1,$$

since

$$ym - m\sqrt{bd} = m(\sqrt{bd+1} - \sqrt{bd}) = \frac{m}{\sqrt{bd+1} + \sqrt{bd}}.$$

Assume for the moment that $m \leq \frac{\sqrt{17}-1}{2}\sqrt{d/b}$, then we have

$$m \leq \frac{\sqrt{17}-1}{2}\sqrt{d/b} < \frac{\sqrt{17}-1}{2}\sqrt{d/15} < \sqrt{d} < \sqrt{bd+1} + \sqrt{bd}.$$

Therefore, we have $\frac{m}{\sqrt{bd+1} + \sqrt{bd}} - 1 < 0$. With these inequalities at hand, we obtain from inequality (29) the following inequality

$$4d < bm^2 + m\sqrt{bd} \frac{(\sqrt{17}-1)^2}{4}d + \frac{\sqrt{17}-1}{2}d = 4d,$$

which is impossible. Hence, we must have $m > \frac{\sqrt{17}-1}{2}\sqrt{d/b}$. \square

Combining all the above lemmas yields the main result of this section.

Lemma 16. *We have $h > (2\sqrt{17}-2)\sqrt{ac} > 6.2462\sqrt{ac}$.*

Proof. Combining Lemma 2, Lemma 13 and Lemma 15, we immediately get

$$\begin{aligned} h &\geq 2m \geq (\sqrt{17}-1)\sqrt{d/b} > (\sqrt{17}-1)\sqrt{4abc/b} \\ &= (2\sqrt{17}-2)\sqrt{ac} > 6.2462\sqrt{ac}. \end{aligned}$$

\square

7. LINEAR FORMS IN LOGARITHMS

In the last section, we found a lower bound for the exponent h . This section is devoted to find good upper bounds for h . This is done by using lower bounds for linear forms in logarithms. To formulate the results concerning lower bounds for linear forms in logarithms, we recall the notation of logarithmic height.

For any non-zero algebraic number γ of degree D over \mathbb{Q} , whose minimal polynomial over \mathbb{Z} is $A \prod_{j=1}^D (X - \gamma^{(j)})$, we denote by

$$h(\gamma) = \frac{1}{D} \left(\log A + \sum_{j=1}^D \log \max \left(1, |\gamma^{(j)}| \right) \right)$$

its absolute logarithmic height. With this notation at hand, we can state the following useful result due to Matveev [36].

Theorem 4. *Let Λ be a linear form in logarithms of N multiplicatively independent totally real algebraic numbers $\alpha_1, \dots, \alpha_N$ with rational integer coefficients b_1, \dots, b_N such that $b_N \neq 0$. Let $h(\alpha_j)$ denote the absolute logarithmic height of α_j , for $1 \leq j \leq N$. Define the numbers D, A_j ($1 \leq j \leq N$) and E by $D := [\mathbb{Q}(\alpha_1, \dots, \alpha_N) : \mathbb{Q}]$, $A_j := \max\{Dh(\alpha_j), |\log \alpha_j|\}$ and $E := \max\{1, \max\{|b_j|A_j/A_N; 1 \leq j \leq N\}\}$. Then,*

$$\log |\Lambda| > -C(N)C_0W_0D^2\Omega,$$

where

$$\begin{aligned} C(N) &:= \frac{8}{(N-1)!}(N+2)(2N+3)(4e(N+1))^{N+1}, \\ C_0 &:= \log(e^{4.4N+7}N^{5.5}D^2\log(eD)), \\ W_0 &:= \log(1.5eED\log(eD)), \quad \Omega := A_1 \cdots A_N. \end{aligned}$$

The main focus of this section will lie on the intersection $X = X_{2h}^{(a,b)} = X_{2j}^{(a,c)}$. From (19) and (20), we obtain recursions for $X_{2h}^{(a,b)}$ and $X_{2j}^{(a,c)}$. By solving these recursions, we obtain

$$\begin{aligned} X_{2h}^{(a,b)} &= \frac{(\sqrt{a} + \sqrt{b})(r + \sqrt{ab})^{2h} - (\sqrt{a} - \sqrt{b})(r - \sqrt{ab})^{2h}}{2\sqrt{b}} \\ (30) \quad X_{2j}^{(a,c)} &= \frac{(\sqrt{a} + \sqrt{c})(s + \sqrt{ac})^{2j} - (\sqrt{a} - \sqrt{c})(s - \sqrt{ac})^{2j}}{2\sqrt{c}}. \end{aligned}$$

This motivates us to define

$$(31) \quad \Lambda_1 = 2h \log(r + \sqrt{ab}) - 2j \log(s + \sqrt{ac}) + \log \left(\frac{\sqrt{c}(\sqrt{a} + \sqrt{b})}{\sqrt{b}(\sqrt{a} + \sqrt{c})} \right).$$

Our first aim is to show that Λ_1 is a rather small, but positive number.

Lemma 17. $0 < \Lambda_1 < (s + \sqrt{ac})^{-4j}$.

Proof. We follow the ideas of Baker and Davenport [2] (see also [15, Lemma 5]). Let

$$P = \frac{(\sqrt{a} + \sqrt{b})(r + \sqrt{ab})^{2h}}{\sqrt{b}}, \quad Q = \frac{(\sqrt{a} + \sqrt{c})(s + \sqrt{ac})^{2j}}{\sqrt{c}}.$$

Using the explicit formulas (30) and substituting P and Q in the right way, we can rewrite the equation $X_{2h}^{(a,b)} = X_{2j}^{(a,c)}$ to

$$P + \frac{b-a}{b}P^{-1} = Q + \frac{c-a}{c}Q^{-1}.$$

This yields

$$P - Q = \frac{c-a}{c}Q^{-1} - \frac{b-a}{b}P^{-1} > \frac{c-a}{c}(Q^{-1} - P^{-1}) = \frac{c-a}{c} \frac{P-Q}{PQ}.$$

In the case that $P - Q < 0$, the above inequality would yield $1 < \frac{c-a}{c} \frac{1}{PQ} = (1 - \frac{a}{c}) \frac{1}{PQ} < 1$, which is an obvious contradiction. Therefore, we have $P - Q > 0$, hence $\Lambda_1 = \log \frac{P}{Q} > 0$.

On the other hand, we have

$$P - Q < \frac{c-a}{c} Q^{-1} < Q^{-1}$$

and we obtain

$$\Lambda_1 = \log \frac{P}{Q} < \log(1 + Q^{-2}) < Q^{-2} < (s + \sqrt{ac})^{-4j}.$$

□

Now, we apply Theorem 4 to Λ_1 with

$$\begin{aligned} N &= 3, \quad D = 4, \quad b_1 = 2h, \quad b_2 = -2j, \quad b_3 = 1, \\ \alpha_1 &= r + \sqrt{ab}, \quad \alpha_2 = s + \sqrt{ac}, \quad \alpha_3 = \frac{\sqrt{c}(\sqrt{a} + \sqrt{b})}{\sqrt{b}(\sqrt{a} + \sqrt{c})}. \end{aligned}$$

Notice that $X^2 - 2rX + 1 = 0$ is the minimal polynomial of $\alpha_1 = r + \sqrt{ab}$ and that $X^2 - 2sX + 1 = 0$ is the minimal polynomial $\alpha_2 = s + \sqrt{ac}$. Therefore, we get

$$h(\alpha_1) = \frac{1}{2} \log \alpha_1, \quad h(\alpha_2) = \frac{1}{2} \log \alpha_2.$$

Since the absolute values of the conjugates of α_3 which are ≥ 1 are

$$\frac{\sqrt{c}(\sqrt{a} + \sqrt{b})}{\sqrt{b}(\sqrt{a} + \sqrt{c})}, \quad \frac{\sqrt{c}(\sqrt{a} + \sqrt{b})}{\sqrt{b}(-\sqrt{a} + \sqrt{c})},$$

and since the minimal polynomial of α_3 is

$$b^2(c-a)^2 X^4 - 4b^2 c(c-a) X^3 + 2bc(3bc - a^2 - ab - ac) X^2 - 4bc^2(b-a) X + c^2(b-a)^2,$$

we obtain that

$$h(\alpha_3) = \frac{1}{4} \log \left(b^2(c-a)^2 \cdot \frac{c}{b} \cdot \frac{(\sqrt{a} + \sqrt{b})^2}{c-a} \right) < \log c.$$

Thus, we choose

$$A_1 = 2 \log \alpha_1, \quad A_2 = 2 \log \alpha_2, \quad A_3 = 4 \log c.$$

Next, we compute the quantity E . By the definition of Λ_1 and Lemma 17, we have $|b_3|A_3 < |b_1|A_1 < |b_2|A_2$. Indeed since $\Lambda_1 > 0$, we deduce that $|b_1|A_1 < |b_2|A_2$. As $4 \log c < 2j \log(2\sqrt{ac})$ and $j \geq 1$, we have $|b_3|A_3 < |b_1|A_1$. Therefore, we get

$$E = \frac{|b_2|A_2}{A_3} = \frac{j \log \alpha_2}{\log c} < \frac{h \log \alpha_3 + \frac{1}{2} \log \alpha_1}{\log c} \leq h.$$

The last inequality can be seen by showing that $\log \alpha_3 + \frac{1}{2} \log \alpha_1 \leq \log c$. Indeed, we have

$$\begin{aligned}
& (\sqrt{ab+1} + \sqrt{ab}) \sqrt{\frac{\sqrt{c}(\sqrt{a} + \sqrt{b})}{\sqrt{b}(\sqrt{a} + \sqrt{c})}} \\
&= (\sqrt{ab+1} + \sqrt{ab}) \sqrt{1 + \frac{\sqrt{ac} - \sqrt{ab}}{\sqrt{bc} + \sqrt{ab}}} \\
&\leq (\sqrt{ab+1} + \sqrt{ab}) \sqrt{1 + \sqrt{\frac{a}{b}}} \\
&\leq (\sqrt{ab+1} + \sqrt{ab}) \left(1 + \frac{1}{2} \sqrt{\frac{a}{b}}\right) \quad (\text{Bernoulli's inequality}) \\
&= \frac{a}{2} + \frac{a}{2} \sqrt{1 + \frac{1}{ab}} + \sqrt{ab+1} + \sqrt{ab} \\
&\leq \frac{a}{2} + \frac{a}{2} \left(1 + \frac{1}{2ab}\right) + \sqrt{ab+1} + \sqrt{ab} \quad (\text{Bernoulli's inequality}) \\
&\leq a + b + 2\sqrt{ab+1} \leq c.
\end{aligned}$$

Before we may apply Theorem 4, we also have to ensure that α_1 , α_2 and α_3 are multiplicatively independent.

Lemma 18. *With the notations above, the algebraic numbers α_1 , α_2 and α_3 are multiplicatively independent.*

Proof. First, we note that α_1 and α_2 are units in the fields $\mathbb{Q}(\sqrt{ab})$ and $\mathbb{Q}(\sqrt{ac})$, respectively. Since a^2bc is not a perfect square, these two fields are linearly disjoint over \mathbb{Q} hence α_1 and α_2 are multiplicatively independent. Furthermore, computing the norm of α_3 , we obtain

$$N_{\mathbb{K}/\mathbb{Q}}(\alpha_3) = \frac{c^2(b-a)^2}{b^2(c-a)^2} \neq \pm 1,$$

where $\mathbb{K} = \mathbb{Q}(\sqrt{ab}, \sqrt{ac})$. Hence α_3 is not a unit and therefore α_1 , α_2 and α_3 are indeed multiplicatively independent. \square

Now by an application of Theorem 4, we have

$$(32) \quad \log |\Lambda| > -4.928 \cdot 10^{12} \cdot \log(38.92h) \cdot \log \alpha_1 \cdot \log \alpha_2 \cdot \log c.$$

Combining inequality (32) with Lemma 17, we obtain that

$$4h \log \alpha_1 < 4j \log \alpha_2 < 4.928 \cdot 10^{12} \cdot \log(38.92h) \cdot \log \alpha_1 \cdot \log \alpha_2 \cdot \log c.$$

Therefore, we obtain the inequality

$$\frac{h}{\log(38.92h)} < 1.232 \cdot 10^{12} \cdot \log \alpha_2 \cdot \log c.$$

Since $\alpha_2 = s + \sqrt{ac+1} < 2\sqrt{ac+1}$, we get

$$(33) \quad \frac{h}{\log(38.92h)} < 1.232 \cdot 10^{12} \cdot \log(2\sqrt{ac+1}) \cdot \log c.$$

By Lemma 16, we have $h > 6.2462\sqrt{ac}$. Moreover, $\frac{h}{\log(38.92h)}$ is an increasing function if $h \geq 1$ and we deduce that

$$(34) \quad \sqrt{ac} < 1.98 \cdot 10^{11} \cdot \log(2\sqrt{ac+1}) \cdot \log c \cdot \log(243.11\sqrt{ac}).$$

A straightforward computation gives $ac < 6.18 \cdot 10^{32}$. Inserting this into (33) we get

$$\frac{h}{\log(38.92h)} < 3.577 \cdot 10^{15},$$

which yields $h < 1.55 \cdot 10^{17}$. Moreover, we deduce that $d < 4abc + 4c < 4(ac)^2 + 4ac < 1.53 \cdot 10^{66}$. Summarizing these results yields.

Proposition 4. *Suppose that $\{a, b, c, d, e\}$ is a Diophantine quintuple with $a < b < c < d < e$. Then we have $ac < 6.18 \cdot 10^{32}$, $d < 1.53 \cdot 10^{66}$ and $h < 1.55 \cdot 10^{17}$.*

This upper bound for d (and also for h) is a slight improvement of that obtained by Wu and the first author [40] and it is a little weaker than the bounds obtained in some special cases obtained by Cipu and Trudgian [8].

In order to get a sharper bound, we use this bound together with a powerful tool due to Mignotte [37]. In fact, some slight different versions of the following theorem were used in some papers, cf. Theorem 12.9 of [5], Theorem 3 of [6]. We use the statement of Proposition 5.1 of [37]. One can refer to the results in Section 12 of [5] and get there the details of the proof. The most closed version is Proposition 3.3 in [3].

Theorem 5. *We consider three non-zero algebraic numbers α_1, α_2 and α_3 , which are either all real and > 1 or all complex of modulus one and all $\neq 1$. Moreover, we assume that either the three numbers α_1, α_2 and α_3 are multiplicatively independent, or two of these numbers are multiplicatively independent and the third one is a root of unity. Put*

$$\mathcal{D} = [\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{Q}] / [\mathbb{R}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{R}].$$

We also consider three positive coprime rational integers b_1, b_2, b_3 , and the linear form

$$\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1 - b_3 \log \alpha_3,$$

where the logarithms of the α_i are arbitrary determinations of the logarithm, but which are all real or all purely imaginary.

And we assume also that

$$b_2 |\log \alpha_2| = b_1 |\log \alpha_1| + b_3 |\log \alpha_3| \pm |\Lambda|.$$

We put

$$d_1 = \gcd(b_1, b_2), \quad d_3 = \gcd(b_3, b_2), \quad b_1 = d_1 b'_1, \quad b_2 = d_1 b'_2 = d_3 b''_2, \quad b_3 = d_3 b''_3.$$

Let $\rho \geq \exp(1)$ be a real number. Put $\lambda = \log \rho$. Let a_1, a_2 and a_3 be real number such that

$$a_i \geq \rho |\log \alpha_i| - \log |\alpha_i| + 2\mathcal{D}h(\alpha_i), \quad i = 1, 2, 3,$$

and assume further that

$$\Omega := a_1 a_2 a_3 \geq 2.5 \quad \text{and} \quad A := \min\{a_1, a_2, a_3\} \geq 0.62.$$

Let K , L , and M be positive integers with

$$L \geq 4 + \mathcal{D}, \quad K = \lfloor M\Omega L \rfloor, \quad \text{where } M \geq 3.$$

Let $\chi > 0$ be fixed and ≤ 2 . Define

$$\begin{aligned} c_1 &= \max \left\{ (\chi ML)^{2/3}, \sqrt{2ML/A} \right\}, \\ c_2 &= \max \left\{ 2^{1/3}(ML)^{2/3}, \sqrt{M/AL} \right\}, \\ c_3 &= (6M^2)^{1/3}L, \end{aligned}$$

and then put

$$\begin{aligned} R_1 &= \lfloor c_1 a_2 a_3 \rfloor, & S_1 &= \lfloor c_1 a_1 a_3 \rfloor, & T_1 &= \lfloor c_1 a_1 a_2 \rfloor, \\ R_2 &= \lfloor c_2 a_2 a_3 \rfloor, & S_2 &= \lfloor c_2 a_1 a_3 \rfloor, & T_2 &= \lfloor c_2 a_1 a_2 \rfloor, \\ R_3 &= \lfloor c_3 a_2 a_3 \rfloor, & S_3 &= \lfloor c_3 a_1 a_3 \rfloor, & T_3 &= \lfloor c_3 a_1 a_2 \rfloor. \end{aligned}$$

Let also

$$R = R_1 + R_2 + R_3 + 1, \quad S = S_1 + S_2 + S_3 + 1, \quad T = T_1 + T_2 + T_3 + 1.$$

Define

$$c_0 = \max \left\{ \frac{R}{La_2 a_3}, \frac{S}{La_1 a_3}, \frac{T}{La_1 a_2} \right\}.$$

Finally, assume that

$$\begin{aligned} (35) \quad & \left(\frac{KL}{2} + \frac{L}{4} - 1 - \frac{2K}{3L} \right) \lambda + 2\mathcal{D} \log 1.36 \\ & \geq (\mathcal{D} + 1) \log L + 3gL^2 c_0 \Omega + \mathcal{D}(K - 1) \log \tilde{b} + 2 \log K, \end{aligned}$$

where

$$g = \frac{1}{4} - \frac{K^2 L}{12RST}, \quad b' = \left(\frac{b'_1}{a_2} + \frac{b'_2}{a_1} \right) \left(\frac{b''_3}{a_2} + \frac{b''_2}{a_3} \right), \quad \tilde{b} = \frac{\exp(3)c_0^2 \Omega^2 L^2}{4K^2} \times b'.$$

Then **either**

$$\log |\Lambda| > -(KL + \log(3KL))\lambda,$$

-or (A1): there exist two non-zero rational integers r_0 and s_0 such that

$$r_0 b_2 = s_0 b_1$$

with

$$|r_0| \leq \frac{(R_1 + 1)(T_1 + 1)}{\mathcal{M} - T_1} \quad \text{and} \quad |s_0| \leq \frac{(S_1 + 1)(T_1 + 1)}{\mathcal{M} - T_1},$$

where

$$\begin{aligned} \mathcal{M} &= \max\{R_1 + S_1 + 1, S_1 + T_1 + 1, R_1 + T_1 + 1, \chi\mathcal{V}\}, \\ \mathcal{V} &= \sqrt{(R_1 + 1)(S_1 + 1)(T_1 + 1)}, \end{aligned}$$

or (A2): there exist rational integers r_1 , s_1 , t_1 , and t_2 , with $r_1 s_1 \neq 0$ such that

$$(t_1 b_1 + r_1 b_3) s_1 = r_1 b_2 t_2, \quad \gcd(r_1, t_1) = \gcd(s_1, t_2) = 1,$$

which also satisfy

$$\begin{aligned} |r_1 s_1| &\leq \delta \cdot \frac{(R_1 + 1)(S_1 + 1)}{\mathcal{M} - \max\{R_1, S_1\}}, \\ |s_1 t_1| &\leq \delta \cdot \frac{(S_1 + 1)(T_1 + 1)}{\mathcal{M} - \max\{S_1, T_1\}}, \\ |r_1 t_2| &\leq \delta \cdot \frac{(R_1 + 1)(T_1 + 1)}{\mathcal{M} - \max\{R_1, T_1\}}, \end{aligned}$$

where $\delta = \gcd(r_1, s_1)$. Moreover, when $t_1 = 0$ we can take $r_1 = 1$, and when $t_2 = 0$ we can take $s_1 = 1$.

Remark 7.1. The cases **(A1)** and **(A1)** represent the case **(C3)** of Theorem 2 in [37].

We aim to apply Theorem 5 to

$$\Lambda := -\Lambda_1 = 2j \log(s + \sqrt{ac}) - 2h \log(r + \sqrt{ab}) - \log \left(\frac{\sqrt{c}(\sqrt{a} + \sqrt{b})}{\sqrt{b}(\sqrt{a} + \sqrt{c})} \right).$$

Therefore we go through the theorem step by step. First, let us assume for technical reasons that $c > 10^6$. As in the previous case we take the parameters

$$\begin{aligned} \mathcal{D} &= 4, \quad b_1 = 2h, \quad b_2 = 2j, \quad b_3 = 1, \\ \alpha_1 &= r + \sqrt{ab}, \quad \alpha_2 = s + \sqrt{ac}, \quad \alpha_3 = \frac{\sqrt{c}(\sqrt{a} + \sqrt{b})}{\sqrt{b}(\sqrt{a} + \sqrt{c})}. \end{aligned}$$

As already shown during the proof of Proposition 4, we have

$$h(\alpha_1) = \frac{1}{2} \log \alpha_1, \quad h(\alpha_2) = \frac{1}{2} \log \alpha_2, \quad h(\alpha_3) < \log c.$$

Moreover, let us note that

$$\begin{aligned} \log \alpha_3 &= \log \left(1 + \frac{\sqrt{a}(\sqrt{c} - \sqrt{b})}{\sqrt{b}(\sqrt{a} + \sqrt{c})} \right) \\ &< \log \left(1 + \sqrt{\frac{a}{b}} \right) < \log \left(1 + \sqrt{\frac{1}{3}} \right) < 0.46. \end{aligned}$$

In view of this inequality, we set

$$a_1 = (\rho + 3) \log \alpha_1, \quad a_2 = (\rho + 3) \log \alpha_2, \quad a_3 = 0.46(\rho - 1) + 8 \log c.$$

We make the following choices for our parameters:

$$\chi = 2, \quad L = 625, \quad M = 12.1, \quad \rho = 10.$$

These choices together with the assumption that $c \geq 10^6$ imply

$$A = \min\{a_1, a_2, a_3\} > 13 \log(\sqrt{c}) > 89.8$$

and therefore, we obtain

$$c_1 = 611.59452 \dots, \quad c_2 = 485.42289 \dots, \quad c_3 = 5985.77903 \dots.$$

With these values, we are able to compute

$$\begin{aligned}
R_1 &= \lfloor c_1 a_2 a_3 \rfloor \simeq 63605.83059 \log \alpha_2 (\log c + 0.5175), \\
R_2 &= \lfloor c_2 a_2 a_3 \rfloor \simeq 50483.98119 \log \alpha_2 (\log c + 0.5175), \\
R_3 &= \lfloor c_3 a_2 a_3 \rfloor \simeq 622521.01991 \log \alpha_2 (\log c + 0.5175), \\
S_1 &= \lfloor c_1 a_1 a_3 \rfloor \simeq 63605.83059 \log \alpha_1 (\log c + 0.5175), \\
S_1 &= \lfloor c_2 a_1 a_3 \rfloor \simeq 50483.98119 \log \alpha_1 (\log c + 0.5175), \\
S_1 &= \lfloor c_3 a_1 a_3 \rfloor \simeq 622521.01991 \log \alpha_1 (\log c + 0.5175), \\
T_1 &= \lfloor c_1 a_1 a_2 \rfloor \simeq 103359.47470 \log \alpha_1 \log \alpha_2, \\
T_2 &= \lfloor c_2 a_1 a_2 \rfloor \simeq 82036.46944 \log \alpha_1 \log \alpha_2, \\
T_3 &= \lfloor c_3 a_1 a_2 \rfloor \simeq 1011596.65736 \log \alpha_1 \log \alpha_2,
\end{aligned}$$

where $R_1 \simeq 63605.83059 \log \alpha_2 (\log c + 0.5175)$ should be read as

$$63605.83059 \log \alpha_2 (\log c + 0.5175) - 1 < R_1 < 63605.8306 \log \alpha_2 (\log c + 0.5175)$$

and so on.

Next, we want to find an upper bound for c_0 . Therefore, we note that we have

$$\begin{aligned}
\frac{R}{La_2 a_3} &= \frac{R_1 + R_2 + R_3 + 1}{La_2 a_3} \\
&\leq \frac{c_1 a_2 a_3 + c_2 a_2 a_3 + c_3 a_2 a_3 + 1}{La_2 a_3} \\
&< \frac{c_1 + c_2 + c_3 + 1}{L} < 11.3341.
\end{aligned}$$

Since similar estimates hold for $\frac{S}{La_1 a_3}$ and $\frac{T}{La_1 a_2}$, we obtain

$$c_0 < 11.3341.$$

With the above choices, we have

$$\begin{aligned}
\Omega &= 1352 \log \alpha_1 \log \alpha_2 (\log c + 0.5175), \\
K &\simeq 10224500 \log \alpha_1 \log \alpha_2 (\log c + 0.5175),
\end{aligned}$$

where we interpret $K \simeq 10224500 \log \alpha_1 \log \alpha_2 (\log c + 0.5175)$ as above.

Our next task is to show that inequality (35) is satisfied. Therefore, we split up the inequality into four parts. One part is representing the left hand side of inequality (35) and the other three parts represent $(\mathcal{D} + 1) \log L + 2 \log K$, $3gL^2 c_0 \Omega$ and $\mathcal{D}(K - 1) \log \tilde{b}$ respectively.

(i) As $M\Omega L - 1 < K \leq M\Omega L$, we have

$$\begin{aligned}
&\left(\frac{KL}{2} + \frac{L}{4} - 1 - \frac{2K}{3L} \right) \lambda + 2\mathcal{D} \log 1.36 \\
&> 7.357094 \cdot 10^9 \log \alpha_1 \log \alpha_2 \log c + 3.807296 \cdot 10^9 \log \alpha_1 \log \alpha_2 - 359.7.
\end{aligned}$$

(ii) Using the upper bound for K , we obtain

$$(\mathcal{D} + 1) \log L + 2 \log K < 87.73.$$

(iii) Using the explicit formula for Ω and noting that $g > \frac{1}{4}$, we get

$$3gL^2c_0\Omega < \frac{3}{4}L^2c_0\Omega < 4.4894 \cdot 10^9 \log \alpha_1 \log \alpha_2 \log c + 2.3233 \cdot 10^9 \log \alpha_1 \log \alpha_2.$$

(iv) For the last part we start by estimating b' . First, let us note that since $\frac{j}{\log \alpha_1} < \frac{h+1}{\log \alpha_2}$, we have $\frac{b_2}{a_1} < \frac{2h+2}{a_2}$. Moreover, we have that $2\log \alpha_2 > \log c$, hence $\frac{b_3}{a_2} < \frac{2}{13\log c} < \frac{2}{a_3}$. Also note that $j \leq h$. Finally, let us note that by Proposition 4, we have that $h \leq 1.55 \cdot 10^{17}$. Therefore, we get

$$b' \leq \left(\frac{b_1}{a_2} + \frac{b_2}{a_1} \right) \left(\frac{b_3}{a_2} + \frac{b_2}{a_3} \right) < \frac{(4h+2)(2h+2)}{104 \log \alpha_2 \log c} < 6.324 \cdot 10^{29}.$$

Thus, we get

$$\log \tilde{b} < \log \left(\frac{6.324 \cdot 10^{29} e^3 c_0^2 \Omega^2 L^2}{4K^2} \right) < 70.1024,$$

which establishes the fourth and last part of inequality (35):

$$\begin{aligned} \mathcal{D}(K-1) \log \tilde{b} &< 4M\Omega L \log \tilde{b} \\ &< 2.8671 \cdot 10^9 \log \alpha_1 \log \alpha_2 \log c + 1.4837 \cdot 10^9 \log \alpha_1 \log \alpha_2. \end{aligned}$$

Combining (i)–(iv), we can now easily verify that condition (35) is satisfied.

According to Theorem 5, we either obtain a lower bound for $|\log \Lambda_1|$ or one of the additional cases **(A1)** and **(A2)** holds. First, let us consider the lower bound for $\log |\Lambda_1|$, which is according to Theorem 5

$$\begin{aligned} \log |-\Lambda_1| &> -(KL + \log(3KL))\lambda > -(ML^2\Omega + \log(3ML^2\Omega)) \log \rho \\ &> -1.52656 \cdot 10^{10} \log \alpha_1 \log \alpha_2 \log c. \end{aligned}$$

On the other hand, Lemma 17 implies that $\log |-\Lambda_1| < -4j \log \alpha_2$. Also note that $h \log \alpha_1 < j \log \alpha_2$, hence

$$(36) \quad h < 3.8164 \cdot 10^9 \log \alpha_2 \log c.$$

Before explicitly solving inequality (36), we discuss the other two options **(A1)** and **(A2)** of Theorem 5. We start by computing \mathcal{M} . As we choose $\chi = 2$, we get

$$\mathcal{M} = \chi\mathcal{V} = 2\mathcal{V} = 2\sqrt{(R_1+1)(S_1+1)(T_1+1)}.$$

First, we consider option **(A2)** and compute

$$\begin{aligned} B_1 &:= \frac{(R_1+1)(S_1+1)}{\mathcal{M} - \max\{R_1, S_1\}}, \\ B_2 &:= \frac{(S_1+1)(T_1+1)}{\mathcal{M} - \max\{S_1, T_1\}}, \\ B_3 &:= \frac{(R_1+1)(T_1+1)}{\mathcal{M} - \max\{R_1, T_1\}}. \end{aligned}$$

By our assumptions and Proposition 4, we have $10^6 < ac < 6.18 \cdot 10^{32}$. Moreover, Lemma 5 implies that $b > c^{1/3}$. Thus, we get

$$29.93 \leq 13 \log \alpha_1 = a_1 < a_2 = 13 \log \alpha_2, \quad 114.66 \leq a_3 = 8(\log c + 0.5175).$$

Finally, remind that

$$c_1 a_2 a_3 - 1 < R_1 \leq c_1 a_2 a_3, \quad c_1 a_1 a_3 - 1 < S_1 \leq c_1 a_1 a_3, \quad c_1 a_1 a_2 - 1 < T_1 \leq c_1 a_1 a_2,$$

which imply that

$$\mathcal{M} > 2c_1^{3/2} a_1 a_2 a_3.$$

Therefore, we obtain the following upper bounds:

$$\begin{aligned} B_1 &\leq \frac{(c_1 a_2 a_3 + 1)(c_1 a_1 a_3 + 1)}{2c_1^{3/2} a_1 a_2 a_3 - c_1 a_2 a_3} \\ &= \left(0.5c_1^{1/2} a_3 + \frac{1}{2c_1^{1/2} a_1} \right) \cdot \frac{1 + \frac{1}{c_1 a_2 a_3}}{1 - \frac{1}{2c_1^{1/2} a_1}} \leq 102.734 \log c, \\ B_2 &\leq \frac{(c_1 a_1 a_3 + 1)(c_1 a_1 a_2 + 1)}{2c_1^{3/2} a_1 a_2 a_3 - c_1 a_1 a_3} \\ &= \left(0.5c_1^{1/2} a_1 + \frac{1}{2c_1^{1/2} a_2} \right) \cdot \frac{1 + \frac{1}{c_1 a_1 a_3}}{1 - \frac{1}{2c_1^{1/2} a_2}} \\ &< \left(0.5c_1^{1/2} a_2 + \frac{1}{2c_1^{1/2} a_2} \right) \cdot \frac{1 + \frac{1}{c_1 a_1 a_3}}{1 - \frac{1}{2c_1^{1/2} a_2}} \leq 160.814 \log \alpha_2, \\ B_3 &\leq \frac{(c_1 a_2 a_3 + 1)(c_1 a_1 a_2 + 1)}{2c_1^{3/2} a_1 a_2 a_3 - c_1 a_2 a_3} \\ &= \left(0.5c_1^{1/2} a_2 + \frac{1}{2c_1^{1/2} a_1} \right) \cdot \frac{1 + \frac{1}{c_1 a_2 a_3}}{1 - \frac{1}{2c_1^{1/2} a_1}} \leq 160.915 \log \alpha_2. \end{aligned}$$

Since we assume that condition **(A2)** holds, there exist rational integers r_1, s_1, t_1 , and t_2 , with $r_1 s_1 \neq 0$ such that

$$(t_1 b_1 + r_1 b_3) s_1 = r_1 b_2 t_2, \quad \gcd(r_1, t_1) = \gcd(s_1, t_2) = 1,$$

with

$$|r_1 s_1| \leq \delta B_1, \quad |s_1 t_1| \leq \delta B_2, \quad |r_1 t_2| \leq \delta B_3, \quad \delta = \gcd(r_1, s_1).$$

Put $r_1 = \delta r'_1$ and $s_1 = \delta s'_1$. As $b_1 = 2h, b_2 = 2j, b_3 = 1$ option **(A2)** is

$$s'_1 t_1 \cdot 2h + \delta r'_1 s'_1 = r'_1 t_2 \cdot 2j,$$

with

$$|\delta r'_1 s'_1| \leq B_1, \quad |s'_1 t_1| \leq B_2, \quad |r'_1 t_2| \leq B_3.$$

Multiplying Λ_1 by $r'_1 t_2$, we obtain the following linear form

$$(37) \quad r'_1 t_2 \Lambda_1 = 2h \log(\alpha_1^{r'_1 t_2} \cdot \alpha_2^{-s'_1 t_1}) - \log(\alpha_2^{\delta r'_1 s'_1} \cdot \alpha_3^{-r'_1 t_2}).$$

Thus, option **(A2)** yields a new linear form in two logarithms. To find a good lower bound for the new linear form (37), we apply a result due to Laurent [34].

Theorem 6 (Theorem 2 of [34]). *Let a'_1, a'_2, h', ϱ , and μ be real numbers with $\varrho > 1$ and $1/3 \leq \mu \leq 1$. Set*

$$\sigma = \frac{1 + 2\mu - \mu^2}{2}, \quad \lambda' = \sigma \log \varrho, \quad H = \frac{h'}{\lambda'} + \frac{1}{\sigma},$$

$$\omega = 2 \left(1 + \sqrt{1 + \frac{1}{4H^2}} \right), \quad \theta = \sqrt{1 + \frac{1}{4H^2}} + \frac{1}{2H}.$$

Consider the linear form

$$\Lambda = b_2 \log \gamma_2 - b_1 \log \gamma_1,$$

where b_1 and b_2 are positive integers. Suppose that γ_1 and γ_2 are multiplicatively independent. Put $D = [\mathbb{Q}(\gamma_1, \gamma_2) : \mathbb{Q}] / [\mathbb{R}(\gamma_1, \gamma_2) : \mathbb{R}]$, and assume that

$$h' \geq \max \left\{ D \left(\log \left(\frac{b_1}{a'_2} + \frac{b_2}{a'_1} \right) + \log \lambda' + 1.75 \right) + 0.06, \lambda', \frac{D \log 2}{2} \right\},$$

$$a'_i \geq \max \{1, \varrho |\log \gamma_i| - \log |\gamma_i| + 2Dh(\gamma_i)\} \quad (i = 1, 2),$$

$$a'_1 a'_2 \geq \lambda'^2.$$

Then

$$\log |\Lambda| \geq -C \left(h' + \frac{\lambda'}{\sigma} \right)^2 a'_1 a'_2 - \sqrt{\omega \theta} \left(h' + \frac{\lambda'}{\sigma} \right) - \log \left(C' \left(h' + \frac{\lambda'}{\sigma} \right)^2 a'_1 a'_2 \right)$$

with

$$C = \frac{\mu}{\lambda'^3 \sigma} \left(\frac{\omega}{6} + \frac{1}{2} \sqrt{\frac{\omega^2}{9} + \frac{8\lambda' \omega^{5/4} \theta^{1/4}}{3\sqrt{a'_1 a'_2} H^{1/2}}} + \frac{4}{3} \left(\frac{1}{a'_1} + \frac{1}{a'_2} \right) \frac{\lambda' \omega}{H} \right)^2, \quad C' = \sqrt{\frac{C \sigma \omega \theta}{\lambda'^3 \mu}}.$$

In order to apply Theorem 6 to $-r'_1 t_2 \Lambda_1$, we consider

$$D = 4, \quad b_1 = 1, \quad b_2 = 2h, \quad \gamma_1 = \alpha_2^{\delta r'_1 s'_1} \cdot \alpha_3^{-r'_1 t_2}, \quad \gamma_2 = \alpha_1^{r'_1 t_2} \cdot \alpha_2^{-s'_1 t_1}.$$

Since α_1, α_2 and α_3 are multiplicatively independent due to Lemma 18 also γ_1 and γ_2 are multiplicatively independent and we may apply Theorem 6.

Also note that a result coming from option **(A2)**, which surpasses (36) would not effect the final result. Therefore, we may assume that h is large, i.e. we may assume that $h \geq 3.8164 \cdot 10^9 \log \alpha_2 \log c$. We will keep this assumption for the rest of the study of option **(A2)**.

Next we have to compute the heights and absolute values of the logarithms of γ_1 and γ_2 . We obtain

$$\begin{aligned} h(\gamma_1) &\leq |\delta r'_1 s'_1| h(\alpha_2) + |r'_1 t_2| h(\alpha_3) \\ &\leq 0.5B_1 \log \alpha_2 + B_3 \log c \leq 212.2811 \log \alpha_2 \log c, \\ h(\gamma_2) &\leq |r'_1 t_2| h(\alpha_1) + |s'_1 t_1| h(\alpha_2) \\ &< 0.5B_2 \log \alpha_1 + 0.5B_3 \log \alpha_2 \leq 160.8641 (\log \alpha_2)^2. \\ |\log \gamma_1| &\leq |\delta s'_1 r'_1| \log \alpha_2 + |t_2 r'_1| \log \alpha_3 \\ &\leq B_1 \log \alpha_2 + 0.46B_3 \leq 102.794 \log \alpha_2 \log c. \end{aligned}$$

In order to get a sharp upper bound for $|\log \gamma_2|$ we have a closer look on the linear form in logarithms (37). Since $|\Lambda_1| < 1$, we obtain the inequality

$$|\log \gamma_2| < \frac{B_3 + |\log(\gamma_1)|}{2h} < 1.5 \cdot 10^{-8},$$

by the previous estimates for B_3 and $|\log(\gamma_1)|$ together with our assumption on h .

In our next step we consider the quantities a'_1 and a'_2 . We choose these quantities such that

$$\begin{aligned} a'_1 &\geq 102.794(\varrho + 1) \log \alpha_2 \log c + 1698.2488 \log \alpha_2 \log c \\ &> (\varrho + 1)|\log \gamma_1| + 8h(\gamma_1) \\ a'_2 &\geq 1.5 \cdot 10^{-8}(\varrho + 1) + 1286.9128(\log \alpha_2)^2 \\ &> (\varrho + 1)|\log \gamma_2| + 8h(\gamma_2). \end{aligned}$$

If we choose $\varrho = 52$ and $\mu = 0.61$, then we get $\sigma = 0.92395$ and $\lambda' = 3.65075 \dots < 3.651$. In view of these choices, we take

$$a'_1 = 7146.331 \log \alpha_2 \log c, \quad a'_2 = 1286.913(\log \alpha_2)^2.$$

If we introduce the quantity

$$F := \frac{2.798639 \cdot 10^{-4}h}{\log \alpha_2 \log c} \geq \frac{b_1}{a'_2} + \frac{b_2}{a'_1}$$

then we can write

$$h' := 4 \log F + 12.2398.$$

On the other hand, we assume that $h \geq 3.8164 \cdot 10^9 \log \alpha_2 \log c$, which yields $F > 1.068 \cdot 10^6$. Thus, we may assume that

$$H = \frac{h'}{\lambda'} + \frac{1}{\sigma} > 19.6429.$$

The lower bound for H gives us now the following upper bounds:

$$\omega < 4.00065, \quad \theta < 1.02578, \quad C < 0.02413, \quad C' < 0.05551.$$

Now we have computed all quantities to apply Theorem 6 and we get

$$\begin{aligned} \log |r'_1 t_2 \Lambda_1| &> -221916.53 \left(h' + \frac{\lambda'}{\sigma} \right)^2 (\log \alpha_2)^3 \log c \\ &\quad - 2.0258 \left(h' + \frac{\lambda'}{\sigma} \right) - \log \left(510509.17 \left(h' + \frac{\lambda'}{\sigma} \right)^2 (\log \alpha_2)^3 \log c \right) \\ &> -221916.6 \left(h' + \frac{\lambda'}{\sigma} \right)^2 (\log \alpha_2)^3 \log c. \end{aligned}$$

By Lemma 17, we have

$$\log |r'_1 t_2 \Lambda_1| < \log B_3 - 4j \log \alpha_2 < \log B_3 - 4h \log \alpha_1$$

and since $\log \alpha_2 < 3 \log \alpha_1$, we get

$$\begin{aligned} h &< 166437.45 \left(h' + \frac{\lambda'}{\sigma} \right)^2 (\log \alpha_2)^2 \log c + \frac{\log(160.915 \log \alpha_2)}{\log \alpha_1} \\ &< 166437.46 \left(h' + \frac{\lambda'}{\sigma} \right)^2 (\log \alpha_2)^2 \log c. \end{aligned}$$

Multiplying the above inequality by $\frac{2.798639 \cdot 10^{-4}}{\log \alpha_2 \log c}$ and noting that $h' = 4 \log F + 12.2398$ we obtain

$$(38) \quad F < 745.278 (\log F + 4.048)^2 (\log \alpha_2).$$

By Proposition 4, we have $ac < 6.18 \cdot 10^{32}$ and so $\log \alpha_2 < 38.446$. From (38) we deduce that

$$F < 28652.96 (\log F + 4.048)^2,$$

which yields $F < 1.18493 \cdot 10^7$. Thus we get the inequality

$$(39) \quad h < 4.234 \cdot 10^{10} \log \alpha_2 \log c.$$

We use the inequality $h > 6.2462\sqrt{ac}$ (cf. Lemma 16) and obtain from inequality (39) that $ac < 1.6 \cdot 10^{26}$. This implies that $\log \alpha_2 < 30.8618$. Inserting this value again into (38) we get

$$F < 23000.63 (\log F + 4.048)^2$$

and we get $F < 9.2851 \cdot 10^6$. Thus, we obtain a slightly improved version of inequality (39)

$$(40) \quad h < 3.3178 \cdot 10^{10} \log \alpha_2 \log c,$$

which is unfortunately still weaker than inequality (36). However, we obtain from this last inequality that $ac < 9.45 \cdot 10^{25}$ and $h < 6.08 \cdot 10^{13}$.

Now, let us briefly discuss option **(A1)**. In this case, we similarly proceed as in the case **(A2)**. However, in this case we apply Theorem 6 to

$$r_0 \Lambda_1 = 2h \log (\alpha_1^{r_0} \alpha_2^{-s_0}) + r_0 \log \alpha_3,$$

with $|r_0| \leq B_3$ and $|s_0| \leq B_2$. Therefore, we choose $\gamma_1 = \alpha_1^{r_0}$ and $\gamma_2 = \alpha_3^{r_0}$. Computing the values for $h(\gamma_i)$ and $|\log \gamma_i|$ for $i = 1, 2$ one obtains smaller values than in the case **(A2)**. Therefore, in the case **(A1)** one obtains smaller upper bounds than indicated by (40).

We apply Theorem 5 in combination with Theorem 6 two times more. Choosing $\rho = 9$, $\chi = 2$, $L = 519$, $M = 14.02$ in Theorem 5 and $\varrho = 57$ and $\mu = 0.61$ in Theorem 6 together with the upper bounds $ac < 9.47 \cdot 10^{25}$ and $h < 6.08 \cdot 10^{13}$ yields the slightly better bounds $ac < 6.87 \cdot 10^{25}$ and $h = 5.18 \cdot 10^{13}$. Now choosing $\rho = 9$, $\chi = 2$, $L = 518$, $M = 13.92$ in Theorem 5 and $\varrho = 56$ and $\mu = 0.61$ in Theorem 6 yields.

Proposition 5. *If $\{a, b, c, d, e\}$ is a Diophantine quintuple with $a < b < c < d < e$, then we have $ac < 6.77 \cdot 10^{25}$, $d < 1.83 \cdot 10^{52}$. Provided that $c > 10^6$, we also have*

$$(41) \quad h < 2.8376 \cdot 10^{10} \log \alpha_2 \log c < 5.136 \cdot 10^{13}.$$

From a computational point of view these upper bounds are still too large to apply the Baker-Davenport reduction method directly.

8. EULER TRIPLES

In this section we still assume that $\{a, b, c, d, e\}$ is a Diophantine quintuple with $a < b < c < d < e$. However, we additionally assume that $\{a, b, c, d\}$ is an Euler quadruples, i.e. a Diophantine quadruple of the form $\{a, b, a+b+2r, 4r(a+r)(b+r)\}$. Particularly, we have

$$s = a + r, \quad t = b + r,$$

and

$$x = at + rs, \quad y = rt + bs, \quad z = cr + st.$$

If we insert the relations for s and t into the relations for x , y and z , then we obtain

$$\begin{aligned} x &= at + rs = ab + 2ar + r^2 = 2r^2 + 2ar - 1 \\ &= 2rs - 1, \\ y &= rt + bs = ab + 2rb + r^2 = 2r^2 + 2rb - 1 \\ &= 2rt - 1, \\ z &= cr + st = ar + br + 2r^2 + st = ab + ar + br + r^2 + st + 1 \\ &= (a + r)(b + r) + st + 1 = 2st + 1. \end{aligned}$$

For the rest of this section, we will use these relations without any special reference. In the next lemma, we will use these relations to obtain several congruence relations that will be crucial in this section.

Lemma 19. *Assume that $\{a, b, c, d, e\}$ is a Diophantine quintuple such that $\{a, b, c\}$ is an Euler triple. Then we have*

$$\begin{aligned} l &\equiv \frac{1 - (-1)^j}{2}(-\varepsilon c) \pmod{s}, & n &\equiv \frac{1 - (-1)^j}{2}(\varepsilon a) \pmod{s}, \\ m &\equiv \frac{1 - (-1)^k}{2}(-\varepsilon c) \pmod{t}, & n &\equiv \frac{1 - (-1)^k}{2}(\varepsilon b) \pmod{t}. \end{aligned}$$

Proof. If we consider relation (20) modulo $2s$, then we obtain

$$\begin{aligned} Z\sqrt{a} + X\sqrt{c} &= (\sqrt{a} + \sqrt{c})(s + \sqrt{ac})^{2j} \\ &= (\sqrt{a} + \sqrt{c})(2s^2 - 1 + 2s\sqrt{ac})^j \\ &\equiv (\sqrt{a} + \sqrt{c})(-1)^j \\ &= (-1)^j\sqrt{a} + (-1)^j\sqrt{c} \pmod{2s}. \end{aligned} \tag{42}$$

We immediately get

$$X = X_{2j}^{(a,c)} \equiv (-1)^j \pmod{2s} \tag{43}$$

and

$$Z = Z_{2j}^{(a,c)} \equiv (-1)^j \pmod{2s}. \tag{44}$$

To obtain further congruences for X and Z , we consider (22) and (24) modulo $2d$ and obtain

$$\begin{aligned}
 (45) \quad W\sqrt{a} + X\sqrt{d} &= (\varepsilon\sqrt{a} + \sqrt{d})(x + \sqrt{ad})^{2l} \\
 &= (\varepsilon\sqrt{a} + \sqrt{d})(2ad + 1 + 2x\sqrt{ad})^l \\
 &\equiv (\varepsilon\sqrt{a} + \sqrt{d})(1 + 2x\sqrt{ad})^l \\
 &\equiv (\varepsilon\sqrt{a} + \sqrt{d})(1 + 2lx\sqrt{ad}) \\
 &\equiv \varepsilon\sqrt{a} + (1 + 2\varepsilon axl)\sqrt{d} \pmod{2d}
 \end{aligned}$$

and

$$\begin{aligned}
 (46) \quad W\sqrt{c} + Z\sqrt{d} &= (\varepsilon\sqrt{c} + \sqrt{d})(z + \sqrt{cd})^{2n} \\
 &= (\varepsilon\sqrt{c} + \sqrt{d})(2cd + 1 + 2z\sqrt{cd})^n \\
 &\equiv (\varepsilon\sqrt{c} + \sqrt{d})(1 + 2z\sqrt{cd})^n \\
 &\equiv (\varepsilon\sqrt{c} + \sqrt{d})(1 + 2nz\sqrt{cd}) \\
 &\equiv \varepsilon\sqrt{c} + (1 + 2\varepsilon czn)\sqrt{d} \pmod{2d},
 \end{aligned}$$

respectively. Therefore, we get some further congruences for X and Z

$$(47) \quad X = X_{2l}^{(a,d)} \equiv 1 + 2\varepsilon axl \pmod{2d}$$

and

$$(48) \quad Z = Z_{2n}^{(c,d)} \equiv 1 + 2\varepsilon czn \pmod{2d}.$$

Since $d = 4rst$ we take (47) modulo $2s$ and combining this congruence with (43) we get

$$1 + 2\varepsilon axl \equiv (-1)^j \pmod{2s},$$

which implies

$$\varepsilon axl \equiv \frac{(-1)^j - 1}{2} \pmod{s}.$$

As $ac \equiv -1 \pmod{s}$, we have

$$xl \equiv \frac{1 - (-1)^j}{2}(\varepsilon c) \pmod{s}.$$

Since $x = 2rs - 1 \equiv -1 \pmod{s}$, we finally get

$$l \equiv \frac{1 - (-1)^j}{2}(-\varepsilon c) \pmod{s},$$

which is the first congruence of Lemma 19.

Similarly, we consider the congruences (44) and (48) and obtain

$$1 + 2\varepsilon czn \equiv (-1)^j \pmod{2s}.$$

Thus, we deduce

$$\varepsilon czn \equiv \frac{(-1)^j - 1}{2} \pmod{s}.$$

Again, as $ac \equiv -1 \pmod{s}$, we get

$$zn \equiv \frac{1 - (-1)^j}{2}(\varepsilon a) \pmod{s}$$

and since $z = 2st + 1 \equiv 1 \pmod{s}$, we have

$$n \equiv \frac{1 - (-1)^j}{2}(\varepsilon a) \pmod{s},$$

which yields the second congruence of Lemma 19. Therefore, we have achieved the first half of Lemma 19.

For the proof of the second half of Lemma 19 we consider (21) modulo $2t$ and get

$$Z\sqrt{b} + Y\sqrt{c} \equiv (-1)^k\sqrt{b} + (-1)^k\sqrt{c} \pmod{2t}.$$

From the above congruence, we obtain two congruences for Y and Z :

$$(49) \quad Y = Y_{2k}^{(b,c)} \equiv (-1)^k \pmod{2t}$$

and

$$(50) \quad Z = Z_{2k}^{(b,c)} \equiv (-1)^k \pmod{2t}.$$

Similarly to (47), we get

$$(51) \quad Y = Y_{2m}^{(b,d)} \equiv 1 + 2\varepsilon bym \pmod{2d}.$$

Combining (49) and (51), we deduce by similar arguments as above that

$$1 + 2\varepsilon bym \equiv (-1)^k \pmod{2t}.$$

Since $bc \equiv -1 \pmod{t}$ and $y \equiv -1 \pmod{t}$, we obtain

$$m \equiv \frac{1 - (-1)^k}{2}(-\varepsilon c) \pmod{t},$$

which establishes the third congruence of Lemma 19.

Finally, we consider the congruences (48) and (50) and obtain

$$1 + 2\varepsilon czn \equiv (-1)^k \pmod{2t}.$$

Using the congruences $bc \equiv -1 \pmod{t}$ and $z \equiv 1 \pmod{t}$, we have

$$n \equiv \frac{1 - (-1)^k}{2}(\varepsilon b) \pmod{t}.$$

This completes the proof of Lemma 19. \square

Lemma 20. *Suppose that $\{a, b, c, d, e\}$ is a Diophantine quintuple such that $\{a, b, c\}$ is an Euler triple. Then, at least one of the following congruences holds*

- I** : $l \equiv n \equiv 0 \pmod{s}$,
- II** : $m \equiv n \equiv 0 \pmod{t}$,
- III**: $n \equiv -\varepsilon r \pmod{st}$.

Proof. If j is even, then $\frac{1-(-1)^j}{2} = 0$ and by Lemma 19, we have

$$l \equiv n \equiv 0 \pmod{s},$$

which yields case I.

If k is even, then we similarly get

$$m \equiv n \equiv 0 \pmod{t},$$

which yields case II.

Therefore, we may assume that both j and k are odd. Thus, Lemma 19 provides us with

$$n \equiv \varepsilon a \pmod{s}, \quad n \equiv \varepsilon b \pmod{t}.$$

Since $s = a + r$ and $t = b + r$, we have

$$n \equiv -\varepsilon r \pmod{s}, \quad n \equiv -\varepsilon r \pmod{t}.$$

As

$$\gcd(s, t) = \gcd(s, s + t) = \gcd(s, c) = 1,$$

we have by the Chinese remainder theorem

$$n \equiv -\varepsilon r \pmod{st}.$$

Therefore, Lemma 20 is proved completely. \square

Our next aim is to show that the options I and II of Lemma 20 yield only small solutions. Such a result will be achieved by using linear forms in logarithms. In particular, let

$$\begin{aligned} \beta_1 &= x + \sqrt{ad}, & \beta_2 &= y + \sqrt{bd}, & \beta_3 &= z + \sqrt{cd}, \\ \beta_4 &= \frac{\sqrt{c}(\varepsilon\sqrt{a} + \sqrt{d})}{\sqrt{a}(\varepsilon\sqrt{c} + \sqrt{d})}, & \beta_5 &= \frac{\sqrt{c}(\varepsilon\sqrt{b} + \sqrt{d})}{\sqrt{b}(\varepsilon\sqrt{c} + \sqrt{d})}. \end{aligned}$$

Then, we consider the two linear forms in logarithms

$$\Lambda_2 = 2l \log \beta_1 - 2n \log \beta_3 + \log \beta_4,$$

and

$$\Lambda_3 = 2m \log \beta_2 - 2n \log \beta_3 + \log \beta_5.$$

As a first step to obtain upper bounds for s and t in case I and II we will consider the linear forms Λ_2 and Λ_3 in case I and II respectively.

First, let us establish good upper bounds for these linear forms in logarithms. This can be done with the help of [15, Lemma 5]. Instead of considering the Diophantine quadruple $\{a, b, c, d\}$, we consider the Diophantine quadruples $\{a, b, d, e\}$ and $\{b, c, d, e\}$ respectively. Finally, note that the condition $c > 4b$ in [15, Lemma 5] is in our situation fulfilled since $d > 4abc > 4b$ and $d > 4abc > 4c$ respectively. Therefore, we obtain the following lemma.

Lemma 21. *We have $0 < \Lambda_2 < \frac{8}{3}ad \cdot \beta_1^{-4l}$ and $0 < \Lambda_3 < \frac{8}{3}bd \cdot \beta_2^{-4m}$.*

The next lemma will deal with the case I.

Lemma 22. *If $l \equiv n \equiv 0 \pmod{s}$, then $s < 20493$.*

Proof. In the case that $r \leq 10000$ we get

$$s = r + a < 2r \leq 20000.$$

Therefore, we may assume that $r > 10000$. Let $l = sl_1, n = sn_1$, for some positive integers l_1, n_1 . We rewrite Λ_2 into the form

$$\Lambda_2 = \log \beta_4 - 2s \log \left(\beta_3^{n_1} / \beta_1^{l_1} \right).$$

In view of an application of Laurent's result Theorem 6 we set

$$D = 4, \quad b_1 = 2s, \quad b_2 = 1, \quad \gamma_1 = \beta_3^{n_1} / \beta_1^{l_1}, \quad \gamma_2 = \beta_4.$$

Next we want to estimate the height of γ_1 . Since β_1 and β_3 are units in $\mathbb{Q}(\sqrt{ad})$ and $\mathbb{Q}(\sqrt{bd})$ respectively, we deduce that γ_1 is an algebraic integer, i.e. the leading coefficient of its minimal polynomial is 1, and the conjugates of γ_1 are

$$\frac{\beta_3^{n_1}}{\beta_1^{l_1}}, \quad \frac{\beta_3^{-n_1}}{\beta_1^{l_1}}, \quad \frac{\beta_3^{n_1}}{\beta_1^{-l_1}}, \quad \frac{\beta_3^{-n_1}}{\beta_1^{-l_1}}.$$

Depending on whether $\beta_3^{n_1} > \beta_1^{l_1}$ or $\beta_3^{n_1} < \beta_1^{l_1}$ we have

$$h(\gamma_1) = \frac{1}{4} \left(\left| \log \frac{\beta_3^{n_1}}{\beta_1^{l_1}} \right| + \left| \log \frac{\beta_3^{n_1}}{\beta_1^{-l_1}} \right| \right) = \frac{n_1}{2} \log \beta_3$$

or

$$h(\gamma_1) = \frac{1}{4} \left(\left| \log \frac{\beta_3^{-n_1}}{\beta_1^{-l_1}} \right| + \left| \log \frac{\beta_3^{-n_1}}{\beta_1^{l_1}} \right| \right) = \frac{l_1}{2} \log \beta_1$$

respectively. On the other hand, the definition of Λ_2 together with Lemma 21 yields

$$\left| \log \left(\beta_3^{n_1} / \beta_1^{l_1} \right) \right| < \frac{1}{s} \left(\log \beta_4 + \frac{8}{3} ad \beta_1^{-4} \right) < \frac{1}{s} \left(\log \beta_4 + \frac{1}{ad} \right).$$

Claiming that $\beta_4 < 2\sqrt{\frac{c}{a}}$, we obtain

$$\left| \log \left(\beta_3^{n_1} / \beta_1^{l_1} \right) \right| < \frac{\log \left(2\sqrt{c/a} \right)}{s} + \frac{1}{sad} < \frac{\log 2s}{s} + \frac{1}{sad} < 0.001$$

since we assume that $s > r > 10000$. Thus, in any case, we have

$$h(\gamma_1) < \frac{l_1}{2} \log \beta_1 + 0.001.$$

To justify our claim that $\beta_4 < 2\sqrt{\frac{c}{a}}$, we compute

$$\begin{aligned} \beta_4 &= \sqrt{\frac{c}{a} \frac{\varepsilon \sqrt{a} + \sqrt{d}}{\varepsilon \sqrt{c} + \sqrt{d}}} = \sqrt{\frac{c}{a}} \left(1 - \varepsilon \frac{\sqrt{c} - \sqrt{a}}{\sqrt{d} + \varepsilon \sqrt{c}} \right) \\ &\leq \sqrt{\frac{c}{a}} \left(1 + \frac{\sqrt{c}}{\sqrt{d} - \sqrt{c}} \right) \leq \sqrt{\frac{c}{a}} \left(1 + \frac{2\sqrt{c}}{\sqrt{d}} \right) \quad \text{since } d > 4abc > 4c \\ &\leq 2\sqrt{\frac{c}{a}}. \end{aligned}$$

Next, we compute the height of $\gamma_2 = \beta_4$. All the absolute values of conjugates of β_4 , namely

$$\frac{\sqrt{c}(\sqrt{d} + \sqrt{a})}{\sqrt{a}(\sqrt{d} + \sqrt{c})}, \quad \frac{\sqrt{c}(\sqrt{d} + \sqrt{a})}{\sqrt{a}(\sqrt{d} - \sqrt{c})}, \quad \frac{\sqrt{c}(\sqrt{d} - \sqrt{a})}{\sqrt{a}(\sqrt{d} + \sqrt{c})}, \quad \frac{\sqrt{c}(\sqrt{d} - \sqrt{a})}{\sqrt{a}(\sqrt{d} - \sqrt{c})},$$

are greater than one. Moreover, the minimal polynomial of β_4 is

$$a^2(d-c)^2X^4 + 4a^2c(d-c)X^3 - 2ac(d^2 + db + dc - 3ac)X^2 + 4ac^2(d-a)X + c^2(d-a)^2.$$

Note that the minimal polynomial does not depend on ϵ since $\frac{\sqrt{c}(\sqrt{d} + \sqrt{a})}{\sqrt{a}(\sqrt{d} + \sqrt{c})}$ and $\frac{\sqrt{c}(\sqrt{d} - \sqrt{a})}{\sqrt{a}(\sqrt{d} - \sqrt{c})}$ are algebraic conjugate. Thus we obtain

$$h(\gamma_2) = h(\beta_4) = \frac{1}{4} \log \left(a^2(d-c)^2 \cdot \frac{c^2}{a^2} \cdot \frac{(d-a)^2}{(d-c)^2} \right) < \frac{1}{2} \log(cd) < \log \beta_3.$$

Choosing $\varrho = 61$ and $\mu = 0.7$ in Theorem 6 we get $\sigma = 0.955$, $\lambda' = 3.92588... < 3.93$. Moreover, we take

$$\begin{aligned} a'_1 &:= 4l_1 \log \beta_1 + 0.07 \\ &> 0.001(\varrho + 1) + 0.008 + 4l_1 \log \beta_1 \\ &\geq \varrho |\log \gamma_1| - \log |\gamma_1| + 8h(\gamma_1). \end{aligned}$$

Since $c = a + b + 2r < 4b$, we have $d > 4abc > c^2$ and $\beta_3 > 2\sqrt{cd} > 2c^{3/2}$. Therefore we obtain

$$\begin{aligned} a'_2 &:= 28 \log \beta_3 > (\varrho - 1) \log(2^{1/3} \sqrt{c}) + 8 \log \beta_3 \\ &> (\varrho - 1) \log \left(\frac{\sqrt{c}(\sqrt{d} + \epsilon \sqrt{a})}{\sqrt{a}(\sqrt{d} + \epsilon \sqrt{c})} \right) + 8 \log \beta_3 \\ &> \varrho |\log \gamma_2| - \log |\gamma_2| + 8h(\gamma_2). \end{aligned}$$

Note that the second inequality holds since

$$\frac{(\sqrt{d} + \sqrt{a})}{\sqrt{a}(\sqrt{d} + \sqrt{c})} \leq \frac{(\sqrt{d} - \sqrt{a})}{\sqrt{a}(\sqrt{d} - \sqrt{c})} \leq 1 + \frac{\sqrt{c} - \sqrt{a}}{\sqrt{d} - \sqrt{c}} \leq 1 + \frac{\sqrt{c}}{\sqrt{60c} - \sqrt{c}} < \frac{7}{6} < 2^{1/3}.$$

In particular note that $d > 4abc \geq 60c$ since $b \geq 15$ due to Lemma 6.

Since $r > 10000$ we have

$$a'_1 > 4 \log(2\sqrt{ad}) > 4 \log(2\sqrt{4rst}) > 4 \log(4(r^3)^{1/2}) > 60$$

and

$$a'_2 > 28 \log(2\sqrt{cd}) > 28 \log(2(16r^4)^{1/2}) > 574.$$

It is easy to see that $a'_1 a'_2 > \lambda'^2$ and our choice of parameters is admissible.

We set

$$b' := \frac{2s}{a'_2} + 0.02 > \frac{2s}{a'_2} + \frac{1}{4l_1 \log \beta_3 + 0.07} = \frac{b_1}{a'_2} + \frac{b_2}{a'_1}$$

and choose

$$\begin{aligned} h' &= 4 \log b' + 12.6 \\ &\geq \max \left\{ D \left(\log \left(\frac{b_1}{a_2'} + \frac{b_2}{a_1'} \right) + \log \lambda' + 1.75 \right) + 0.06, \lambda', \frac{D \log 2}{2} \right\}. \end{aligned}$$

Therefore, as $\beta_3 = z + \sqrt{cd} < 2z$ and $z = 2st + 1 < 2s^3 + 1$, we have

$$\begin{aligned} h' &= 4 \log \left(\frac{s}{14 \log \beta_3} + 0.02 \right) + 12.6 \\ &> 4 \log \left(\frac{s}{14 \log(4s^3 + 2)} \right) + 12.6 > 25.4. \end{aligned}$$

This implies that $H > 7.5$. Then, we have $\omega < 4.01$, $\theta < 1.07$. And hence

$$B := \frac{1}{4} \left(h' + \frac{\lambda'}{\sigma} \right) < \log b' + 4.2.$$

We obtain

$$C < 0.0226, \quad C' < 0.047.$$

Now, we apply Theorem 6 and get

$$(52) \quad \log |\Lambda_1| \geq -0.3616B^2 a_1' a_2' - 8.29B - \log(0.76B^2 a_1' a_2').$$

By Lemma 21, we have

$$(53) \quad \log |\Lambda_1| < -4sl_1 \log \beta_1 + \log \left(\frac{8}{3} ad \right) = -s(a_1' - 0.07) + \log \left(\frac{8}{3} ad \right).$$

Combining (52) and (53), we obtain

$$(54) \quad s(a_1' - 0.07) < 0.3616B^2 a_1' a_2' + 8.29B + \log(0.76B^2 a_1' a_2') + \log \left(\frac{8}{3} ad \right).$$

Dividing both sides of the above inequality by $a_1' a_2'$ and simplifying, we obtain

$$b' < 0.725(\log b' + 4.2)^2.$$

Thus, we get $b' < 46.98$. As $b' = \frac{2s}{a_2'} + 0.02$, we deduce that

$$s < 23.48a_2' = 23.48(28 \log \beta_3 + 0.04) < 657.44 \log(4s^3 + 2) + 1.$$

Therefore, we obtain $s < 20493$. □

Lemma 23. *If $m \equiv n \equiv 0 \pmod{t}$, then $t < 22023$.*

Proof. The proof is similar to that of Lemma 22. However we may only assume that $r \geq 145$. Indeed assuming $r < 145$ yields $t = r + b < r + r^2 < 21170$. Note that assuming $r \leq 150$ would result in $t \leq 22650$ which already succeeds the bound given in the Lemma.

By the assumption of the lemma we have that $m \equiv n \equiv 0 \pmod{t}$, and we may write $m = tm_2, n = tn_2$ for some positive integers m_2, n_2 . We rewrite Λ_3 into the form

$$\Lambda_3 = \log \beta_5 - 2t \log(\beta_3^{n_2} / \beta_2^{m_2})$$

and apply Theorem 6 to this linear form. As the application of Theorem 6 is technically similar to that in the proof of Lemma 22, we omit the details. We

only want to note that the slightly larger upper bound is due to the fact that we only assume that $r \geq 145$. Hence we obtain smaller lower bounds for a'_1, a'_2, h', H and so on. Therefore, we obtain slightly larger upper bounds for C, C' and so on resulting in a slightly larger upper bound for t . \square

Next, we consider case III

Lemma 24. *If $n \equiv -\varepsilon r \pmod{st}$, then we have $r < 900154$ and $h < 9.6 \cdot 10^{15}$.*

Proof. By Lemma 11 and Lemma 15, we have

$$n \geq \frac{m}{2} \geq \frac{1}{2} \cdot \frac{\sqrt{17}-1}{2} \cdot \sqrt{\frac{d}{b}} \geq \frac{\sqrt{17}-1}{2} \sqrt{ac} > s > r.$$

If $n \equiv -\varepsilon r \pmod{st}$, then $n + \varepsilon r \geq st$, hence $n \geq st - r \geq c(r-1)$. On the other hand, we have by Lemma 11 and Lemma 13 that $h \geq 2n$ and therefore, we get

$$h \geq 2c(r-1).$$

In view of the statement of the lemma we may assume that $r > 900000$. This implies that $c = a + b + 2r > 10^6$. Due to Proposition 5, we have

$$h < 2.8376 \cdot 10^{10} \log(s + \sqrt{ac}) \log c.$$

Combining the upper and lower bound for h , we have

$$c(r-1) \leq 1.4188 \cdot 10^{10} \log(2\sqrt{c(r-1)}) \log(c(r-1)/900000).$$

This implies that

$$c(r-1) < 3.233 \cdot 10^{12}.$$

Since $c = a + b + 2r \geq 2\sqrt{ab} + 2r > 3.99r$ we have

$$3.99r(r-1) < 3.233 \cdot 10^{12}$$

and therefore, we obtain $r < 900154$.

We are left to compute the upper bound for h . If $c > 10^6$, then by Proposition 5 we have

$$\begin{aligned} h &< 2.8376 \cdot 10^{10} \log(s + \sqrt{ac}) \log c < 2.8376 \cdot 10^{10} \log(2(a+r)) \log(a+b+2r) \\ &< 2.8376 \cdot 10^{10} \log(4r-2) \log(1+r^2+2r) < 1.2 \cdot 10^{13}. \end{aligned}$$

If $c \leq 10^6$, then we get the following inequalities coming from inequality (33)

$$\frac{h}{\log(38.92h)} < 1.232 \cdot 10^{12} \cdot \log(2\sqrt{ac+1}) \cdot \log c < 2.36 \cdot 10^{14}$$

and so $h < 9.6 \cdot 10^{15}$. \square

Now, assume that $\{a, b, c, d, e\}$ is a Diophantine quintuples such that $a < b < c < d < e$ and $\{a, b, c\}$ is an Euler triple. Then, by Lemma 20 and Lemmas 22–24, we have

$$r < 900154, \quad \text{and} \quad h < 9.6 \cdot 10^{15}.$$

Note that the upper bounds obtained from Lemma 22 and Lemma 23 are much smaller than the bound obtained from Lemma 24.

In order to deal with the remaining cases, we will use a Diophantine approximation algorithm called the Baker-Davenport reduction method. The following

lemma is a slight modification of the original version of the Baker-Davenport reduction method (see [19, Lemma 5a]).

Lemma 25. *Assume that M is a positive integer. Let p/q be the convergent of the continued fraction expansion of a real number κ such that $q > 6M$ and let*

$$\eta = \|\mu q\| - M \cdot \|\kappa q\|,$$

where $\|\cdot\|$ denotes the distance from the nearest integer. If $\eta > 0$, then the inequality

$$0 < J\kappa - K + \mu < AB^{-J}$$

has no solutions in integers m and l with

$$\frac{\log(Aq/\eta)}{\log B} \leq J \leq M.$$

We apply Lemma 25 to

$$\Lambda_1 = 2h \log(r + \sqrt{ab}) - 2j \log(s + \sqrt{ac}) + \log \left(\frac{\sqrt{c}(\sqrt{a} + \sqrt{b})}{\sqrt{b}(\sqrt{a} + \sqrt{c})} \right).$$

with $s = a + r$, $c = a + b + 2r$ and

$$\kappa = \frac{\log(r + \sqrt{ab})}{\log(s + \sqrt{ac})}, \quad \mu = \frac{\log \left(\frac{\sqrt{c}(\sqrt{a} + \sqrt{b})}{\sqrt{b}(\sqrt{a} + \sqrt{c})} \right)}{\log(s + \sqrt{ac})},$$

$$A = \frac{1}{\log(s + \sqrt{ac})}, \quad B = (r + \sqrt{ab})^2$$

and $J = 2h$, $M = 9.6 \cdot 10^{15}$. We ran a GP program to check all 58258307 pairs (a, b) such that $2 \leq r \leq 900153$ and obtained $J \leq 15$ in each case. This contradicts the fact that $J = 2h \geq 4c(r - 1) \geq 48$. It took 7 hours and 3 minutes to run the program on a MacBook Pro with an i7 4960hq CPU and 16G memory. Summarizing our results we obtain:

Theorem 7. *An Euler triple $\{a, b, a + b + 2\sqrt{ab + 1}\}$ cannot be extended to a Diophantine quintuple.*

9. NON-EULER TRIPLES

In this section, we will deal with the non-Euler triples. We will consider two cases: the case that the degree is one and the case that the degree is greater than one. We start with the case that the degree is one.

Theorem 8. *A Diophantine triple $\{a, b, c\}$ cannot be extended to a Diophantine quintuple if $\deg(a, b, c) = 1$.*

Proof. Assume that $a < b < c$. If $\deg(a, b, c) = 1$, then the triple $\{d_{-1}, a, b\}$ is an Euler triple, where

$$d_{-1} = d_-(a, b, c)$$

Moreover the quadruple $\{d_{-1}, a, b, c\}$ is regular due to Proposition 2. Therefore we have

$$d_{-1} = a + b \pm 2r$$

and

$$c = d_+(a, d_{-1}, b) = 4r(r \pm a)(b \pm r).$$

We assume that $b > 10000$. Then, $d_{-1} \geq a + b - 2\sqrt{\frac{b^2}{24} + 1} > 0.59b$ and together with the observation that $c > 4abd_{-1}$, we have

$$2.36(ab)^2 < 4a^2bd_{-1} < ac < 6.77 \cdot 10^{25}.$$

Hence, we get $ab < 5.36 \cdot 10^{12}$ and $r \leq 2315167$. Moreover, we have $a < \left(\frac{r^2}{2}\right)^{\frac{2}{5}} < 93596$ since $b > \max\{24a, 2a^{3/2}\}$ due to Lemma 3.

We apply Lemma 25 to Λ_1 and check 109748916 pairs (a, r) such that

$$b = (r^2 - 1)/a, \quad c = 4r(r \pm a)(b \pm r),$$

and κ, μ, A, B are taken as in the previous section. Moreover, we choose $J = 2h$ and $M = 9.6 \cdot 10^{15}$. The running time of the GP program is less than 16 hours. In all 219497932 cases, we have $J \leq 15$. This contradicts the fact that $J = 2h > 10\sqrt{ac} > 20\sqrt{2}$. \square

Now we have to deal with the case that the degree of the triple is greater than one. In this case we prove the following theorem.

Theorem 9. *A Diophantine triple $\{a, b, c\}$ cannot be extended to a Diophantine quintuple if $\deg(a, b, c) \geq 2$.*

Proof. Let us consider a Diophantine triple $\{a, b, c\}$ with $\deg(a, b, c) \geq 2$. Using Proposition 5, we may assume that $ac < 6.77 \cdot 10^{25}$. Since $\deg(a, b, c) \geq 2$, there exist positive integers d_{-1} and d_{-2} satisfying

$$d_{-1} = d_-(a, b, c), \quad d_{-2} = d_-(a, b, d_{-1}).$$

By Lemma 1, since $\{a, b, c\}$ is not an Euler triple we have $c > a + b + 2\sqrt{ab} + 1$ and therefore $c > 4ab$. Furthermore, we have $ac < 0.927b^3$ due to Lemma 5. These observations imply

$$4ab < c < \frac{b^3}{a}.$$

For the remaining proof, we split up the interval $\left(4ab, \frac{b^3}{a}\right)$ into four subintervals:

$$c \in \left(4ab, 4a^{\frac{1}{2}}b^{\frac{3}{2}}\right] \cup \left(4a^{\frac{1}{2}}b^{\frac{3}{2}}, 4ab^2\right] \cup \left(4ab^2, 4ab^{\frac{5}{2}}\right] \cup \left(4a^{\frac{3}{2}}b^{\frac{5}{2}}, b^3/a\right).$$

• **Case I:** $4ab < c \leq 4a^{\frac{1}{2}}b^{\frac{3}{2}}$. Since $c = d_+(a, b, d_{-1})$, we have that $c > 4abd_{-1}$ and therefore, we obtain

$$ad_{-1} < \frac{c}{4b} < (ab)^{\frac{1}{2}},$$

and in particular we have that $ab > (ad_{-1})^2$. Since $c > 4abd_{-1}$ we get $ac > 4(ab)(ad_{-1})$ and moreover

$$ad_{-1} < \left(\frac{ac}{4}\right)^{\frac{1}{3}} < \left(\frac{6.77 \cdot 10^{25}}{4}\right)^{\frac{1}{3}} < 256749472.$$

Put $r_{(a, d_{-1})} = \sqrt{ad_{-1} + 1}$. Since $ad_{-1} + 1$ is a perfect square, $r_{(a, d_{-1})}$ is a positive integer with $2 \leq r_{(a, d_{-1})} \leq 16023$. Using a short GP program, we see

that there are 1081908 pairs (a, d_{-1}) to be checked in this range. Note that $\{a, d_{-1}, b\}$ is a Diophantine triple. For a fixed pair (a, d_{-1}) , there exist positive integers $U = r = \sqrt{ab+1}$ and $V = \sqrt{bd_{-1}+1}$ such that $b = \frac{V^2-1}{d_{-1}} = \frac{U^2-1}{a}$ and $\max\{U, V\} \leq b^{\frac{1}{2}}$. Indeed, we have

$$4a^2b < a \cdot 4ab < ac < 6.77 \cdot 10^{25}$$

and therefore we may assume that $\max\{U, V\} \leq 4.12 \cdot 10^{12}$.

In order to find all possible values of b , we consider the Pell equation

$$(55) \quad \mathcal{A}\mathcal{V}^2 - \mathcal{B}\mathcal{U}^2 = \mathcal{A} - \mathcal{B},$$

where $\mathcal{A}\mathcal{B} + 1 = \mathcal{R}^2$, $0 < \mathcal{A}, \mathcal{B}, \mathcal{R} \in \mathbb{Z}$ and $\mathcal{A} < \mathcal{B}$. By Lemma 7, all positive integer solutions to the above Pell equation can be determined by

$$\mathcal{V}\sqrt{\mathcal{A}} + \mathcal{U}\sqrt{\mathcal{B}} = \mathcal{V}_q\sqrt{\mathcal{A}} + \mathcal{U}_q\sqrt{\mathcal{B}} = (\mathcal{V}_0\sqrt{\mathcal{A}} + \mathcal{U}_0\sqrt{\mathcal{B}})(\mathcal{R} + \sqrt{\mathcal{A}\mathcal{B}})^q, \quad q \geq 0,$$

where $(\mathcal{V}_0, \mathcal{U}_0)$ is a fundamental solution to Pell equation (55), hence satisfies

$$0 \leq |\mathcal{V}_0| \leq \sqrt{\frac{1}{2}\mathcal{A}(\mathcal{B} - \mathcal{A})(\mathcal{R} - 1)}, \quad 0 < \mathcal{U}_0 \leq \sqrt{\frac{\mathcal{A}(\mathcal{B} - \mathcal{A})}{2(\mathcal{R} - 1)}}.$$

Our program runs over all $r_{(a, d_{-1})}$ in the range from 2 to 16023. For each $\mathcal{R} = r_{(a, d_{-1})}$, we consider the divisors d' of $\mathcal{R}^2 - 1$ with $1 \leq d' \leq \mathcal{R}$ and put $\mathcal{A} = d'$, $\mathcal{B} = (\mathcal{R}^2 - 1)/\mathcal{A}$. For each pair $(\mathcal{A}, \mathcal{B})$, we find all possible fundamental solutions $(\mathcal{V}_0, \mathcal{U}_0)$ to equation (55) and consider the corresponding sequences \mathcal{U}_q . Notice that not all solutions \mathcal{U} of (55) satisfy $\mathcal{A} | (\mathcal{U}^2 - 1)$. If $\mathcal{A} | (\mathcal{U}^2 - 1)$ and $\mathcal{U} = \mathcal{U}_q < 4.12 \cdot 10^{12}$, then we put (a, d_{-1}, b) or $(d_{-1}, a, b) = (\mathcal{A}, \mathcal{B}, \mathcal{C})$ and $c = d_+(a, d_{-1}, b)$, where $\mathcal{C} = \frac{\mathcal{U}^2 - 1}{\mathcal{A}}$.

Applying Lemma 25 to Λ_1 , we checked all 2340242 triples possible (a, b, c) in 15 minutes with our GP program. In all cases we obtain that $J \leq 5$, which is impossible as $J > 20\sqrt{2}$.

• **Case II :** $4a^{\frac{1}{2}}b^{\frac{3}{2}} < c < 4ab^2$. Since $c = d_+(a, b, d_{-1})$, we deduce from our assumption that $d_{-1} < \frac{c}{4ab} < b$. This implies that $b = \max\{a, b, d_{-1}\}$ and Lemma 2 yields $c < 4b(ad_{-1} + 1)$. On the other hand, we have by our assumption that $4a^{\frac{1}{2}}b^{\frac{3}{2}} < c$ and therefore we get $(ab)^{\frac{1}{2}} - 1 < ad_{-1}$. Thus, we get

$$d_{-2} = d_-(a, b, d_{-1}) < \frac{b}{4ad_{-1}} < \frac{b}{4((ab)^{\frac{1}{2}} - 1)}.$$

and

$$4ad_{-2} < \frac{ab}{(ab)^{\frac{1}{2}} - 1} < (ab)^{\frac{1}{2}} + 2.$$

From $(4ad_{-2} - 2)^2 < ab < (ad_{-1} + 1)^2$ we have $4ad_{-2} < ad_{-1} + 3$ and so $d_{-2} < d_{-1}$. Substituting $ab > (4ad_{-2} - 2)^2$ into the inequality $ac > 4(ab)(ad_{-1}) > 4(ab)(ad_{-2})$, we obtain

$$4(4ad_{-2} - 2)^2(ad_{-2}) < ac < 6.77 \cdot 10^{25}.$$

It follows that $ad_{-2} < 101891096$ and we get that $r_{(a,d_{-2})} = \sqrt{ad_{-2} + 1} \leq 10095$. Moreover, we know that

$$d_{-1} < b < \left(\frac{6.77 \cdot 10^{25}}{4} \right)^{2/3} < 6.6 \cdot 10^{16}.$$

Similarly to Case I, we put (a, d_{-2}) or $(d_{-2}, a) = (\mathcal{A}, \mathcal{B})$ in equation (55). And we set (a, d_{-2}, d_{-1}) or $(d_{-2}, a, d_{-1}) = (\mathcal{A}, \mathcal{B}, (\mathcal{U}_q^2 - 1)/\mathcal{A})$ when $\mathcal{A}|\mathcal{U}_q^2 - 1$, for $\mathcal{U}_q < b^{1/2} < 2.57 \cdot 10^8$. Using $b = d_+(a, d_{-1}, d_{-2})$ and $c = d_+(a, b, d_{-1})$, we apply Lemma 25 to Λ_1 and check 2565234 triples (a, b, c) . The verification with our GP program took 20 minutes and we obtained that $J \leq 6$ in all cases. But this is also impossible as $J > 20\sqrt{2}$.

• **Case III:** $4ab^2 < c < 4a^{\frac{3}{2}}b^{\frac{5}{2}}$. First, we observe that the inequality $4a^2b^2 < ac < 6.77 \cdot 10^{25}$ yields the upper bound $r < 2028300$. Since $b^{1/2} < \sqrt{ab+1} = r$, we also have an upper bound for $b^{1/2}$.

Assume for the moment that $b > d_{-1}$, then $b = d_+(a, d_{-1}, d_{-2}) > 4ad_{-1}$ and therefore $d_{-1} < \frac{b}{4a}$. But this yields

$$4ab^2 < c < 4abd_{-1} + 4b < b^2 + 4b,$$

which is impossible. Therefore, we may assume that $b < d_{-1}$. Since $d_{-1} < \frac{c}{4ab}$, we get

$$d_{-1} < a^{\frac{1}{2}}b^{\frac{3}{2}}.$$

and therefore

$$ad_{-2} < \frac{d_{-1}}{4b} < \frac{(ab)^{\frac{1}{2}}}{4} < \frac{r}{4} < 507075.$$

If we write $r_{(a,d_{-2})} = \sqrt{ad_{-2} + 1}$, then we have that $r_{(a,d_{-2})} \leq 713$. Using the algorithm of Case I, we set (a, d_{-2}) or $(d_{-2}, a) = (\mathcal{A}, \mathcal{B})$ in equation (55). We take (a, d_{-2}, b) or $(d_{-2}, a, b) = (\mathcal{A}, \mathcal{B}, (\mathcal{U}_q^2 - 1)/\mathcal{A})$ when $\mathcal{A}|\mathcal{U}_q^2 - 1$, for $\mathcal{U}_q < b^{1/2} < 2028300$. Using $d_{-1} = d_+(a, d_{-2}, b)$ and $c = d_+(a, b, d_{-1})$, we apply Lemma 25 to Λ_1 and check all 98746 triples (a, b, c) with our GP program, which took 2 minutes. In all cases we obtain $J \leq 15$. This contradicts the fact that $J > 20\sqrt{2}$.

• **Case IV:** $4a^{\frac{3}{2}}b^{\frac{5}{2}} < c < \frac{b^3}{a}$. In this case, we have $b > 16a^5$. Therefore, we get from

$$4096a^{15} = 4(16a^6)^{\frac{5}{2}} < 4(ab)^{\frac{5}{2}} < ac < 6.77 \cdot 10^{25}$$

that $a \leq 30$. By the same arguments as in Case III we may assume that $b < d_{-1}$. Moreover, we have

$$d_{-1} < \frac{c}{4ab} < \frac{b^2}{4a^2} \quad \text{and} \quad d_{-2} < \frac{d_{-1}}{4ab} < \frac{b}{16a^3}.$$

On the other hand, $4(ab)^{5/2} < ac < 6.77 \cdot 10^{25}$ yields $ab < 1.24 \cdot 10^{10}$. Therefore, we get

$$16a^4d_{-2} < 1.24 \cdot 10^{10}.$$

Let us write $r_{(a,d_{-2})} = \sqrt{ad_{-2} + 1}$, then we get that $r_{(a,d_{-2})} \leq \frac{27839}{a^{3/2}}$. There are 42362 pairs (a, d_{-2}) satisfying this inequality and $a \leq 30$. We set (a, d_{-2}) or

$(d_{-2}, a) = (\mathcal{A}, \mathcal{B})$ in equation (55). For solutions \mathcal{U}_q to Pell equation (55) we put (a, d_{-2}, b) or $(d_{-2}, a, b) = (\mathcal{A}, \mathcal{B}, (\mathcal{U}_q^2 - 1)/\mathcal{A})$, if $\mathcal{A}|\mathcal{U}_q^2 - 1$. Note that we may assume that $\mathcal{U}_q < b^{1/2} < 111356$ since $b \leq ab < 1.24 \cdot 10^{10}$. Furthermore we compute $d_{-1} = d_+(a, d_{-2}, b)$ and $c = d_+(a, b, d_{-1})$ and apply Lemma 25 to Λ_1 . Our GP program checked all 151076 triples (a, b, c) in 5 minutes and we got $J \leq 15$ in each case. This also contradicts the fact $J > 20\sqrt{2}$.

This completes the proof of Theorem 9. \square

10. PROOF OF THEOREM 1

Assume that $\{a, b, c, d, e\}$ is a Diophantine quintuple, with $a < b < c < d < e$. By Theorem 3 (cf. [25]), the quadruple $\{a, b, c, d\}$ is regular, i.e. the element d is uniquely determined by a, b and c . By Proposition 3, for an arbitrary but fixed Diophantine triple $\{a, b, c\}$, there exists a nonnegative integer $D = \deg(a, b, c)$ such that $\{a, b, c\}$ is generated by some Euler triple $\{a', b', c'\}$. Theorem 7, Theorem 8 and Theorem 9 show that a Diophantine triple $\{a, b, c\}$ cannot be extended to Diophantine quintuple $\{a, b, c, d, e\}$ for $D = 0, 1$ and $D \geq 2$, respectively. This completes the proof of Theorem 1.

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DEPARTMENT OF MATHEMATICS, HUBEI UNIVERSITY FOR NATIONALITIES, ENSHI, HUBEI,
445000 P.R. CHINA AND INSTITUTE OF MATHEMATICS, ABA TEACHERS UNIVERSITY, WENCHUAN,
SICHUAN, 623000 P. R. CHINA

E-mail address: `bhe@live.cn`

DEPARTMENT OF MATHEMATICS, STATISTICS, AND COMPUTER SCIENCE PURDUE UNIVER-
SITY NORTHWEST, 1401 S, U.S. 421, WESTVILLE IN 46391 USA

E-mail address: `atogbe@pnw.edu`

INSTITUTE OF MATHEMATICS, UNIVERSITY OF SALZBURG, HELLBRUNNER STRASSE 34/I, A-
5020 SALZBURG, AUSTRIA

E-mail address: `volker.ziegler@sbg.ac.at`